

ELEMENTS

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ALGEBRA.

ABERDEEN:

MDCCLXXVI.

ELEMENTS

These Elements of Algebra were drawn up in order to be used as a Syllabus or Text, in teaching this branch of Science; and for this purpose, simplicity and perspicuous arrangement were principally kept in view.



INTRODUCTION.

QUANTITY which can be measured, and is the object of mathematics, is of two kinds, *Number* and *Extension*. The former is treated of in ARITHMETIC; the latter in GEOMETRY.

Numbers are ranged in a Scale, by the continued repetition of some one Number, which is called the Root; and in consequence of this order, they are conveniently expressed in words, and denoted by characters. The operations of arithmetic are easily derived from the established method of notation, and the most simple reasonings concerning the relations of magnitude.

Investigations by the common arithmetic are greatly limited, from the want of characters to express the quantities that are unknown, and their different relations to one another, and to such as are known. Hence letters and other convenient symbols have been introduced to supply this defect; and thus gradually has arisen the science of ALGEBRA, properly called UNIVERSAL ARITHMETIC.

In the common arithmetic too, the given numbers disappear in the course of the operation, so that general rules can seldom be derived from it; but in algebra, the known quantities as well as the unknown may be expressed by letters, which thro' the whole operation retain their original form; and hence may be deduced, not only general canons for like cases, but the connexion and dependence of the several quantities concerned, and also the determination of a problem, without exhibiting which it is not completely resolved.

If geometrical quantities be supposed to be divided into equal parts, their relations in respect of magnitude, or their proportions, may be expressed by numbers; one of these equal parts being denoted by the unit. Arithmetic however is used in expressing only the conclusions of geometrical propositions, and it is by algebra that the bounds and application of geometry have been of late so far extended.

The proper objects of mathematical science are number and extension; but mathematical enquiries may be instituted also concerning any physical quantities, that are capable of being measured or expressed by numbers and extended magnitudes. And as the application of algebra may be equally universal, it has been called *The science of quantity in general*.

E L E M E N T S

O F

A L G E B R A.

P A R T I.

D E F I N I T I O N S.

- I. **Q**UANTITIES which are known, are generally represented by the first letters of the alphabet, as $a, b, c,$ &c. and such as are unknown, by the last letters, as $x, y, z,$ &c.
- II. The sign $+$ (*plus*) placed before any quantity, denotes that, that quantity is to be added; and when no sign is expressed, $+$ is understood. Thus $a + b$ denotes the sum of a and b ; $3 + 5$ denotes the sum of 3 and 5, or 8.
- III. The sign $-$ (*minus*) denotes that the quantity, before which it is placed, is to be subtracted. Thus $a - b$ denotes the excess of a above b : $6 - 2$ is the excess of 6 above 2, or 4.
- Note,* These characters $+$ and $-$ from their extensive use in Algebra are called *the signs*; and the one is said to be opposite or contrary to the other.
- IV. Quantities which have the sign $+$ prefixed to them, are called *positive* or *affirmative*; and such as have the sign $-$ prefixed to them, are called *negative*.
- V. Quantities which have the same sign, either $+$ or $-$, are also said to have *like signs*, and those which have different signs, are said to have *unlike signs*. Thus, $+a, +b,$ have like signs, and $+a, -c,$ are said to have unlike signs.
- VI. The *juxtaposition* of letters as in the same word, expresses the product of the quantities denoted by these letters.

Thus ab expresses the product of a and b ; bcd expresses the continued product of b , c , and d . The sign $\times$ also expresses the product of any two quantities between which it is placed.

VII. A number prefixed to a letter, is called a *numeral coefficient*, and expresses the product of the quantity by that number, or how often the quantity denoted by the letter is to be taken. When no number is prefixed, unit is understood.

VIII. The *quotient* of two quantities is denoted by placing the *dividend* above a small line, and the *divisor* below it. Thus, $\frac{18}{3}$ is the quotient of 18 divided by 3 or 6; $\frac{a}{b}$ is the quotient of a divided by b . This expression of a quotient is also called a *fraction*.

IX. A quantity is said to be *simple*, which consists of one part or *Term*, as $+a$, $-abc$; and a quantity is said to be *compound*, when it consists of more than one term, connected by the signs $+$ or $-$. Thus, $a+b$, $a-b+c$, are compound quantities. If there are two terms, it is called a *binomial*: if three, a *trinomial*, &c.

X. Simple quantities or the terms of compound quantities are said to be *like*, which consist of the same letter or letters, equally repeated. Thus $+ab$, $-5ab$, are like quantities; but $+ab$, and $+aab$, are unlike.

XI. The equality of two quantities is expressed by placing the sign $=$ between them. Thus $x+a=b-c$, means that the sum of x and a , is equal to the excess of b above c .

When quantities are considered abstractly, then $+$ and $-$ denote addition and subtraction only, according to Def. II. III. and the terms *positive* and *negative* express the same ideas. In that case, a negative quantity by itself is unintelligible. The sign $+$ also is unnecessary before simple quantities, or before the leading term of a compound quantity which is not negative; tho' when such a quantity or term is to be added to another, $+$ must be placed before it to express that addition, and hence in Def. II. it is said that $+$ is understood, when no sign is expressed.

In certain quantities, however, as particularly in Geometrical

metrical, there may be an opposition or contrariety. entirely analogous to that of addition and subtraction; and the signs + and - may be used to express that contrariety. In such cases, negative quantities are understood to exist by themselves; and the same rules take place in operations into which they enter, as are used with regard to the negative terms of abstract quantities.

C H A P. I.

Fundamental Operations.

THE fundamental operations in Algebra are the same as in common arithmetic, **ADDITION, SUBTRACTION, MULTIPLICATION, and DIVISION**; and from the various combinations of these four, all the others are derived.

P R O B. I. *To add Quantities.*

Simple quantities, or the terms of compound quantities, to be added together, may be *like with like signs, like with unlike signs, or they may be unlike.*

CASE I. To add Terms that are like, and have like signs.

Rule. *Add together the coefficients, to their sum prefix the common sign, and subjoin the common letter or letters.*

E X A M P L E S.

To	$5ab$	$3aa-ab$
Add	$4ab$	$7aa-2ab$
		$4aa-5ab$
Sum	$9ab$	
		$14aa-8ab.$

CASE II. To add Terms that are like, but have unlike signs.

Rule. *Subtract the less coefficient from the greater, prefix the sign of the greater to the remainder, and subjoin the common letter or letters.*

E X A M P L E S.

$-4a$	$+7bc$	$-5ab$
$+7a$	$-3bc$	$+2ab$
	$+bc$	$+3ab$
$+3a$		
	$+5bc$	

CASE

CASE III. To add Terms that are unlike.

Rule. Set them all down, one after another, with their signs and coefficients prefixed.

E X A M P L E.

$$\begin{array}{r} 2a + 3b \\ -5c + 8 \\ \hline \end{array}$$

$$2a + 3b - 5c + 8$$

Compound quantities are added together, by uniting the several terms, of which they consist, by the preceeding rules.

E X A M P L E.

$$\begin{array}{l} \text{The sum of } \left\{ \begin{array}{l} 5ab - 3xy - 12cd \\ 7xy - ab + 15 \\ 9cd - 4xy - mn \end{array} \right. \\ \hline \end{array}$$

$$\text{is } 4ab - 3cd + 15 - mn$$

addedy The rule for case III. may be considered as the general rule for dividing all Algebraical Quantities whatsoever, and by the rules in the two preceeding cases the like Terms in the Quantities to be added may be united so as to render the expression of the sum more simple.

P R O B. II. To subtract Quantities.

General Rule. Change the signs of the quantity to be subtracted into the contrary signs, and then add it, so changed, to the quantity from which it was to be subtracted: (by Prob I.) the sum arising by this addition is the remainder.

E X A M P L E S.

$$\begin{array}{r} \text{From} \quad + 5a \\ \text{Subtract} \quad + 3a \\ \hline \end{array} \qquad \begin{array}{r} 7ab - 16bc \\ 3ab + mb \\ \hline \end{array}$$

$$\text{Rem.} \quad + 2a \qquad 4ab - 16bc - mb$$

$$\begin{array}{r} \text{From} \quad 5a - 7b + 9c + 8 \\ \text{Subt.} \quad 2a - 4b + 9c - d \\ \hline \end{array}$$

$$\text{Rem.} \quad 3a - 3b + 8 + d$$

When a positive quantity is to be subtracted, the rule is obvious: in order to shew it, when the negative part of a quantity is to be subtracted, let $c-d$ be subtracted from $a-b$, the remainder, according to the rule, is $a-b-c+d$. For if c is subtracted from $a-b$, the remainder plainly is $a-b-c$; but this is too small, because c is subtracted instead of $c-d$, which

is less than it by d ; the remainder therefore is too small by d ; and d being added, it is $a - b - c + d$, according to the rule.

P R O B. III. To multiply Quantities.

General Rule for the Signs. *When the signs of the two terms to be multiplied are like, the sign of the product is +; but when the signs are unlike, the sign of the product is -.*

CASE I. To multiply two terms.

Rule. Find the sign of the product by the general rule; after it place the product of the numeral coefficients, and then set down all the letters one after another, as in one word.

Mult.	$+a$	$+5b$	$-5ax$
By	$+b$	$-3c$	$-7ab$
	$+ab$	$-15bc$	$+35aabcx$

The reason of this rule is derived from Def. 6. and from the nature of multiplication, which is a repeated addition of one of the quantities to be multiplied as often as there are units in the other. Hence also the letters in two terms multiplied together may be placed in any order, and therefore the order of the alphabet is generally preferred.

CASE II. To multiply compound quantities.

Rule. Multiply every term of the multiplicand, by all the terms of the multiplier, one after another, according to the preceeding rule, and then collect all the products into one sum: That sum is the product required.

E X A M P L E S.

Mult.	$2a+3b$	$m+x$
By	$3ax-4by$	$m-x$
	$6aax+9abx$	$mm+mx$
	$-8aby-12bby$	$-mx-xx$
Prod.	$6aax+9abx-8aby-12bby$	$mm+mx-xx$

Mult.	$a-b$
By	$c-d$
	$ac-cb$
	$-ad+db$
Prod.	$ac-cb-ad+db$

Of

Of the general Rule for the Signs.

The reason of every part of that rule, will appear from the last mentioned example of $a-b$, multiplied by $c-d$.

Since multiplication is a repeated addition of the multiplicand, as often as there are units in the multiplier, hence, if $a-b$ is to be multiplied by c , $a-b$ must be added to itself, as often as there are units in c , and the product must obviously be $ca-cb$. (Prob. I.)

But this product is too great, for $a-b$ is to be multiplied by $c-d$ only, and not by c : therefore d times $a-b$, or $da-db$ is to be subtracted from the former part of the product, and the remainder which (by prob. II.) is $ca-cb-da+db$, will be the true product required.

Def. 12. When several Quantities are multiplied together, any of them is called a factor of the Product.

13. The products arising from the continual multiplication of the same quantity, are called the *Powers* of that quantity, which is the *Root*. Thus aa , aaa , $aaaa$, &c. are powers of the root a .

14. These powers are expressed by placing above the root to the right-hand a figure denoting how often the root is repeated. This figure is called an *index* or *exponent*, and from it the power is denominated. Thus,

a	} is called the	{ 1st	} Power of the Root	{ a^1 or a				
aa					{ 2d	{ a , and is o-		
aaa							{ 3d	{ therwise expres-
$aaaa$								
				{ a^2				
				{ a^3				
				{ a^4 &c.				

The 2d and 3d powers are generally called the square and Cube; and the 4th, 5th, and 6th, are also sometimes respectively called the Biquadrate, Surfsolid, and Cubocube.

Cor. Powers of the same root are multiplied by adding their exponents. Thus, $a^3 \times a^2 = a^5$, or $aaa \times aa = aaaaa$, $b^3 \times b = b^4$.

S C H O L I U M.

Sometimes it is convenient to express the multiplication of quantities, by setting them down with the sign ($\times$) between them, without performing the operation according the preceding rules, thus $a^2 \times b$ is written instead of $a^2 b$. and $\overline{a-b} \times \overline{c-d}$, expresses the product of $a-b$, multiplied by $c-d$.

Def. 15. A *Vinculum* is a line drawn over any number of terms of a compound quantity to denote those which are understood to be affected by the particular sign connected with it,

Thus

Thus in the last example it shews, that the terms $+a$ and $-b$, and also c and $-d$ are all affected by the sign ($\times$). Without the vinculum, the expression $a-b \times c-d$ would mean the excess of a above bc and d ; and $\overline{a-b} \times c-d$ would mean the excess of the Product of $a-b$ by c , above d . Thus also, $\overline{a+b}^2$, expresses the second power of $a+b$, or the product of that quantity multiplied by itself; whereas $a+b^2$ would express only the sum of a and b^2 ; and so of others.

P R O B. VI. *To divide Quantities.*

General rule for the signs. *If the signs of the divisor and dividend are like, the sign of the quotient is +; if they are unlike, the sign of the quotient is -.*

This rule is easily deduced from that given in Prob. 3. for from the nature of division, the quotient must be such a quantity as multiplied by the divisor, shall produce the dividend with its proper sign.

From Def. 8. the Quotient of any two Quantities may be expressed by placing the dividend above a line and the divisor below it. But a quotient may often be expressed in a more simple and convenient form, as will appear from the following distinction of the Cases.

Case I. When the divisor is simple and is a *factor* of all the Terms of the Dividend. This is easily discovered by inspection; for then the coefficient of the divisor measures that of all the Terms of the dividend, and all the letters of the divisor are found in every Term of the Dividend.

Rule. *The letter or letters in the divisor, are to be expunged out of each term in the dividend, and the coefficients of each term to be divided by the coefficient of the divisor; the quantity resulting is the quotient.*

Ex. $a) ab(b. \quad 2aab) \quad 6a^3bc-4a^2bdc \quad (3ac-2dm.$

The reason of this is evident from the nature of division, and from def. 6.

Note. It is obvious from corollary to prob. 3, that powers of the same root are divided by subtracting their exponents.

Thus $a^2)a^3$ ($a. \quad a^3)a^7$ ($a^4. \quad$ also $a^2b)a^3b^6$ ($ab^5.$

CASE II. When the divisor is simple, but not a factor of the dividend.

Rule. *The quotient is expressed by a fraction, according to def. 8. viz. by placing the dividend above a line; and the divisor below it.*

Thus

Thus the quotient of $3ab^2$ divided by $2mbc$ is the fraction

$$\frac{3ab^2}{2mbc}.$$

Such expressions of quotients may often be reduced to a more simple form, as shall be explained in the second part of this chapter.

CASE III. When the divisor is compound.

R U L E.

1. *The terms of the dividend are to be ranged according to the powers of some one of its letters; and those of the divisor, according to the powers of the same letter.*

Thus, if $a^2 + 2ab + b^2$ is the dividend, and $a + b$ the divisor, they are ranged according to the powers of a .

2. *The first term of the dividend is to be divided by the first term of the divisor, (observing the general rule for the signs) and this quotient being set down, as a part of the quotient wanted, is to be multiplied by the whole divisor, and the product subtracted from the dividend. If nothing remain, the division is finished: the remainder, when there is any, is a new dividend.*

Thus in the preceeding example, a^2 divided by a , gives a , which is the first part of the quotient wanted: and the product of this part by the whole divisor $a + b$, viz. $a^2 + ab$ being subtracted from the given dividend, there remains in this example $ab + b^2$.

3. *Divide the first term of this new dividend, by the first term of the divisor as before, and join the quotient to the part already found, with its proper sign: then multiply the whole divisor by this part of the quotient, and subtract the product from the new dividend: and thus the operation is to be continued till no remainder is left, or till it appear, that there will always be a remainder.*

Thus in the preceeding example, $+ab$, the first term of the new dividend divided by a , gives b ; the product of which, multiplied by $a + b$, being subtracted from $ab + b^2$, nothing remains, and $a + b$ is the true quotient.

The

The entire operation is as follows:

$$\begin{array}{r}
 a + b) a^3 + 2ab + b^3 (a + b) \\
 \underline{a^3 + ab} \\
 ab + b^3 \\
 \underline{ab + b^2} \\
 b^3
 \end{array}$$

$$\begin{array}{r}
 a - b) 3a^3 - 12a^2 - a^2b + 10ab - 2b^3 (a^3 - 4a^2 + 2b) \\
 \underline{3a^3} \\
 -12a^2 \qquad + 10ab \\
 \underline{-12a^2} \qquad + 4ab \\
 \qquad \qquad + 6ab - 2b^3 \\
 \qquad \qquad \underline{+ 6ab - 2b^3} \\
 \qquad \qquad \qquad 0
 \end{array}$$

$$\begin{array}{r}
 1 - a) 1 (1 + a + a^2 + a^3, \&c.) \\
 \underline{1 - a} \\
 \qquad + a \\
 \qquad \underline{+ a - a^2} \\
 \qquad \qquad + a^2 \\
 \qquad \qquad \underline{+ a^2 - a^3} \\
 \qquad \qquad \qquad + a^3, \&c.
 \end{array}$$

It often happens, as in the last example, that there is still a remainder from which the operation may be continued without end. This expression of a quotient is called *an infinite series*; the nature of which shall be considered afterwards. By comparing a few of the first terms, the *law* of the series may be discovered, by which, without any more division, it may be continued to any number of terms wanted.

Of the General Rule.

The reason of the different parts of this rule is evident ; for in the course of the operation, all the terms of the quotient obtained by it, are multiplied by all the terms of the divisor, and the products are successively subtracted from the dividend, till nothing remain : that, therefore, from the nature of division, must be the true quotient.

Note. The sign $\div$ is sometimes used to express the quotient of two quantities, between which it is placed : thus

$\frac{a^2 + x^2}{a + x} \div a + x$, expresses the quotient of $a^2 + x^2$ divided by $a + x$.

CHAP. I. PART II.

OF FRACTIONS.

DEFINITIONS.

- I. **W**HEN a quotient is expressed by a fraction, the dividend above the line is called the *numerator*; and the divisor below it is called the *denominator*.
- II. If the numerator is less than the denominator, it is called a *proper fraction*.
- III. If the numerator is not less than the denominator, it is called an *improper fraction*.
- IV. If one part of a quantity is an integer, and the other a fraction, it is called a *mixt quantity*.
- V. The *reciprocal* of a fraction, is a fraction whose numerator is the denominator of the other; and whose denominator is the numerator of the other. The reciprocal of an integer is the quotient of 1 divided by that integer. Thus $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$; and $\frac{1}{m}$ is the reciprocal of m .

The distinctions in Def. II. III. IV. properly belong to common arithmetic, from which they are borrowed, and are scarcely used in Algebra.

The operations concerning fractions are founded on the following proposition :

If the divisor and dividend be either both multiplied, or both divided, by the same quantity, the quotient is the same; or if both the numerator and denominator of the fraction be either multiplied or divided by the same quantity, the value of that fraction is the same.

Thus let $\frac{a}{b} = c$, then $\frac{ma}{mb} = c$. For from the nature of division if the quotient $\frac{a}{b} (= c)$ be multiplied by the divisor

b the product must be the dividend a . Hence $\left(\frac{a}{b} \times b =\right) bc = a$, and likewise $ma = mbc$, and dividing both by mb , $\frac{ma}{mb} = c$.

Conversely if $\frac{ma}{mb} = c$, then also $\frac{a}{b} = c$.

Cor. 1. Hence a fraction may be reduced to another of the same value, but of a more simple form; by dividing both numerator and denominator by any common measure.

$$\begin{array}{r} \text{Thus } 30ax - 54ay = 5x - 9y. \\ \hline 12ab \qquad \qquad 2b \\ 8ab + 6ac = 4b + 3c. \\ \hline 4a^2 \qquad \qquad 2a \end{array}$$

Cor. 2. A fraction is multiplied by any integer, by multiplying the numerator, or dividing the denominator by that integer: and conversely, a fraction is divided by any integer, by dividing the numerator, or multiplying the denominator by that integer.

Prob. I. To find the greatest common measure of two quantities.

1. Of pure numbers.

Rule. Divide the greater by the less: and if there is no remainder, the less is the greatest common measure required. If there is a remainder, divide the last divisor by it; and thus proceed continually dividing the last divisor by its remainder, till no remainder is left, and the last divisor is the greatest common measure required.

The greatest common measure of 45 and 63 is 9; the greatest common measure of 187 and 391 is 17. Thus

$$\begin{array}{r} 45) 63 (1 \\ \underline{45} \\ 18) 45 (2 \\ \underline{36} \\ 9) 18 (2 \\ \underline{18} \\ 0 \end{array}$$

$$\begin{array}{r} 187) 391 (2 \\ \underline{374} \\ 17) 187 (11 \\ \underline{187} \\ 0 \end{array}$$

This rule depends on the two following evident principles, 1st. A quantity which measures both divisor and remainder will also measure the dividend; 2d. A quantity which measures both divisor and dividend must also measure the remainder.

By

By the first it appears that the number found by this rule is a common measure, and by the second it is plain there can be no greater common measure; for if there were, it must necessarily measure the quantity already found less than itself, which is absurd.

When the greatest common measure of algebraical quantities is required, if either of them be simple, any common simple divisor is easily found by inspection. If they are both compound any common simple divisor may also be found by inspection. But when the greatest compound divisor is wanted the preceeding rule is to be applied only,

2. The simple divisors of each of the quantities are to be taken out, the remainders in the several operations are also to be divided by their simple divisors, and the quantities are always to be ranged according to the powers of the same letter.

The simple divisors in the given quantities or in the remainders do not affect a compound divisor which is wanted, and hence also to make the division succeed, any of the dividends may be multiplied by a simple quantity. Besides the simple divisors in the remainders not being found in the divisors from which they arise, can make no part of the common measure sought, and for the same reason, if in such a remainder there be any compound divisor which does not measure the divisor from which it proceeds, it may be taken out.

EXAMPLES.

$$\frac{a^2 - b^2}{a^2 - b^2} (a^2 - 2ab + b^2) (1$$

$$a^2 - b^2$$

divided by $-2ab + 2b^2$ Remainder which
 $-2b$ is $a - b$) $a^2 - b^2$ ($a + b$
 $a^2 - b^2$

$$-2b \text{ is } \frac{a-b}{a^2-b^2} a^2 - b^2 (a+b)$$

$$a^2 - b^2$$

If the quantities given are $8a^2b^2 - 10ab^3 + 2b^4$, and $9a^4b - 9a^3b^2 + 3a^2b^3 - 3ab^4$. The simple divisors being taken out, viz. $2b^2$ out of the first, it becomes $4a^2 - 5ab + b^2$, and $3ab$ out of the second, it is $3a^3 - 3a^2b + ab^2 - b^3$. As the latter is to be divided by the former, it must be multiplied by 4 to make the operation succeed, and then it is as follows.

$$(4a^2 - 5ab + b^2) \begin{pmatrix} 12a^3 - 12a^2b + 4ab^2 - 4b^3 \\ 12a^3 - 15a^2b + 3ab^2 \end{pmatrix} (3a$$

$$12a^3 - 15a^2b + 3ab^2$$

$$3a^2b + ab^2 - 4b^3$$

This

This remainder is to be divided by b , and the new dividend multiplied by 3, to make the division proceed, thus

$$\begin{array}{r} 3a^2 + ab - 4b^2 \) \ 12a^2 - 15ab + 3b^2 \ (\ 4 \\ \underline{12a^2 + \quad 4ab - 16b^2} \end{array}$$

$$- 19ab + 19b^2$$

and this remainder divided by $-19b$ gives $a - b$ which being made a divisor, divides, $3a^2 + ab - 4b^2$ without a remainder, and therefore $a - b$ is the greatest compound divisor: but there is a simple divisor b , and therefore $a - b \times b$ is the greatest common measure required.

Prob. II. To reduce a fraction to its lowest terms.

Rule. Divide both numerator and denominator by their greatest common measure, which may be found by prob. 1.

Thus, $\frac{75abc}{125bcx} = \frac{3a}{5x}$, $25bc$ being the greatest common measure.

$\frac{a^4 - b^4}{a^5 - a^3b^2} = \frac{a^2 + b^2}{a^3}$. also, $\frac{9a^4b - 9a^3b^2 + 3a^2b^3 - 3ab^4}{8a^2b^2 - 10ab^3 + 2b^4} = \frac{9a^3 + 3ab^2}{8ab - 2b^2}$ the greatest common measure being $a - b \times b$, by prob. 1.

Prob. III. To reduce an integer to the form of a fraction.

Rule. Multiply the given integer by any quantity for a numerator, and set that quantity under the product, for a denominator.

$$\text{Thus } a = \frac{ma}{m}, \quad a + b = \frac{a^2 - b^2}{a - b}.$$

Cor. Hence, in the following operations concerning fractions, an integer may be introduced; for, by this problem, it may be reduced to the form of a fraction. The denominator of an integer is generally made 1.

Prob. IV. To reduce fractions with different denominators to fractions of equal value, that shall have the same denominator.

Rule. Multiply each numerator, separately taken, into all the denominators but its own, and the products shall give the new numerators. Then multiply all the denominators into one another, and the product shall give the common denominator.

Example

Example. Let the fractions be $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$; they are respectively equal to $\frac{adf}{bdf}$, $\frac{bcf}{bdf}$, $\frac{bde}{bdf}$.

The reason of the operation appears from the preceeding proposition; for the numerator and denominator of each fraction are multiplied by the same quantities; and the value of the fractions therefore is the same.

Prob. V. To add and subtract fractions.

Rule. Reduce them to a common denominator, then add or subtract the numerators; and the sum or difference set over the common denominator is the sum or remainder required.

Ex. Add together $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, the sum is $\frac{adf + cbf + bde}{bdf}$.

From $\frac{a}{b}$ subtr. $\frac{c}{d}$ the difference is $\frac{ad - bc}{bd}$.

From the nature of division it is evident, that when several quantities are to be divided by the same divisor, the sum of the quotients is the same with the quotient of the sum of the quantities divided by that common divisor.

In like manner the difference of two fractions having the same denominator, is equal to the difference of the numerators divided by that common denominator.

Cor. From Cor. Prob. 3. Integers may be reduced to the form of fractions, and hence integers and fractions may be added and subtracted by this rule. Hence also what is called a mixed quantity may be reduced into the form of a fraction by bringing the integral part into the form of a fraction, with the same denominator as the fractional part, and adding or subtracting the numerators according as the two parts are connected by the signs + or -.

$$\text{Thus, } b + \frac{c}{d} = \frac{bd+c}{d} \text{ and } a - \frac{a^2-b^2}{2a} = \frac{2a^2-a^2+b^2}{2a} \\ = \frac{a^2+b^2}{2a}.$$

Sometimes it is convenient to distinguish a fraction into parts by dividing the numerator, into several parts, and setting each over the original denominator and uniting the new fractions by the signs of their numerators.

Thus

Thus, $\frac{a^2 - 2ab + b^2}{2a} = \frac{a^2}{2a} - \frac{2ab}{2a} + \frac{b^2}{2a} = \frac{a}{2} - b + \frac{b^2}{2a}$.

Prob. VI. To multiply fractions.

Rule. *Multiply their numerators into one another, to obtain the numerator of the product; and the denominators, multiplied into one another, shall give the denominator of the product.*

Ex. $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$. $\frac{a+b}{c} \times \frac{a-b}{d} = \frac{a^2 - b^2}{cd}$.

For if $\frac{a}{b}$ is to be multiplied by c , the product is $\frac{ca}{b}$; but if it is to be multiplied only by $\frac{c}{d}$, the former product must be divided by d , and it becomes $\frac{ca}{bd}$ (cor. 2. to the preceding problem.)

Or let $\frac{a}{b} = m$, and $\frac{c}{d} = n$. Then $a = bm$, and $c = dn$ and $ac = bdmn$, and $(mn) = \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

Prob. VII. To divide fractions.)

Rule. *Multiply the numerator of the dividend by the denominator of the divisor; their product shall give the numerator of the quotient. Then multiply the denominator of the dividend by the numerator of the divisor, and their product shall give the denominator.*

Or, *Multiply the dividend by the reciprocal of the divisor; the product will be the quotient wanted.*

Thus, $\frac{a}{b} \div \frac{c}{d} = \frac{bc}{ad} = \frac{c}{d} \times \frac{b}{a}$.

For if $\frac{c}{d}$ is to be divided by a , the quotient is $\frac{c}{da}$ but $\frac{c}{d}$ is to be divided not by a , but by $\frac{a}{b}$; therefore the former quotient must be multiplied by b , and it is $\frac{bc}{da}$.

Or let $\frac{a}{b} = m$, and $\frac{c}{d} = n$; then $a = bm$, and $c = dn$; also $ad = bdm$ and $bc = bdn$; therefore $\left(\frac{bdn}{bdm}\right) = \frac{n}{m} = \frac{bc}{ad}$.

SCHO.

S C H O L I U M.

By these problems, the four fundamental operations may be performed, when any terms of the original quantities, or of those which arise in the course of the operation, are fractional,

Example

$$\text{Mult. } \frac{a^2}{2x} - \frac{3ax}{2b}$$

$$\text{By } \frac{ab}{3x} - 4x$$

$$\text{Prod. } \frac{a^3b}{6x^2} - \frac{a^2}{2} - 2a^2 + \frac{6ax^2}{b}$$

$$a+x) \frac{a^3 + x^3}{a^2 + ax} (a - x + \frac{2x^2}{a} - \frac{2x^3}{a^2}, \&c.$$

$$\underline{-ax + x^3}$$

$$\underline{-ax - x^3}$$

$$\frac{2x^2}{2x^2} + \frac{2x^3}{a}$$

$$\underline{-\frac{2x^3}{a}}$$

$$\frac{2x^3}{a} - \frac{2x^4}{a^2}$$

$$\underline{+\frac{2x^4}{a^2}}, \&c$$

This quotient becomes a series, of which the law of continuation is obvious, without any farther operation.

In such cases, when we arrive at a remainder of one term, it is commonly set down with the divisor below it, after the other terms of the quotient, which then becomes a mixt quantity.

Thus the last quotient is also expressed by $a - x + \frac{2x^2}{a+x}$.

C H A P. II.

Of Proportion.

BY the preceeding operations, quantities of the same kind may be compared together.

The relation arising from this comparison is called *Ratio* or *Proportion*, and is of two kinds. If we consider the difference of the two quantities, it is called *Arithmetical Proportion*; and if we consider their quotient, it is called *Geometrical Proportion*. This last being most generally useful, is commonly called simply *Proportion*.

1. *Of Arithmetical Proportion.*

Definition. When of four quantities the difference of the first and second is equal to the difference of the third and fourth, the quantities are called *arithmetical proportionals*.

Cor. Three quantities may be arithmetically proportional, by supposing the two middle terms of the four to be equal.

Prop. In four quantities arithmetically proportional, the sum of the extremes is equal to the sum of the means.

Let the four be a, b, c, d . Therefore from Def. $a - b = c - d$; to these add $b + d$ and $a + d = b + c$.

Cor. Of four arithmetical proportionals, any three being given, the fourth may be found.

Thus let a, b, c , be the 1st, 2d, and 4th terms, and let x be the 3d which is sought,

Then by def. $a + c = b + x$, and $x = a + c - b$,

2. *Of Geometrical Proportion.*

Definition. If of four quantities, the quotient of the first and second is equal to the quotient of the third and fourth, these quantities are said to be in *geometrical proportion*. They are also called *proportionals*.

Thus if a, b, c, d , are the four quantities, then $\frac{a}{b} = \frac{c}{d}$. and their ratio is thus denoted, $a : b :: c : d$;

C

Cor.

Cor. Three quantities may be geometrical proportionals, viz. by supposing the two middle terms of the four to be equal. If the quantities are a, b, c , then $\frac{a}{b} = \frac{b}{c}$, and the proportion is expressed thus, $a : b :: b : c$.

Prop. I. The product of the extremes of four quantities geometrically proportional is equal to the product of the means : and conversely.

Let $a : b :: c : d$.

Then by Def. $\frac{a}{b} = \frac{c}{d}$

and multiplying both by bd , $ad = bc$.

If $ad = bc$, then dividing by bd , $\frac{a}{b} = \frac{c}{d}$, that is, $a : b :: c : d$.

Cor. 1. The product of the extremes of three quantities, geometrically proportional, is equal to the square of the middle term.

Cor. 2. Of four quantities geometrically proportional, any three being given, the fourth may be found.

Ex. Let a, b, c , be the three first; to find the 4th. Let it be x , then $a : b :: c : x$, and by this proposition,

$$ax = bc$$

and dividing both by a , $x = \frac{bc}{a}$.

This coincides with the *Rule of Three* in arithmetic, and may be considered as a demonstration of it. In applying the rule to any particular case, it is only to be observed that the quantities must be so connected and so arranged that they be proportional, according to the preceeding definition.

Prop. II. If four quantities be geometrically proportional, then if any equimultiples whatever be taken of the first and third, and also any equimultiples whatever of the second and fourth; if the multiple of the first be greater than that of the second, the multiple of the third will be greater than that of the fourth; and if equal, equal; and if less, less.

For let a, b, c, d , be the four proportionals. Of the first and third, ma and mc may represent any equimultiples whatever, and also, nb, nd , may represent any equimultiples of the second and fourth. Since $a : b :: c : d$; $ad = bc$, and hence multiplying by mn , $mnad = mnbc$, and therefore (Conv. Prop. 1.)

ma

$ma : nb :: mc : nd$; and from the definition of proportionals it is plain that if ma is greater than nb , mc must be greater than nd ; and if equal, equal: and if less, less.

Prop. III. If four quantities are proportionals, they will also be proportionals when taken *alternately* or *inversely*, or by *composition*, or by *division*, or by *conversion*. See def. 13, 14, 15, 16, 17, of book V. of Euclid.

By Prop. II. they will also be proportionals according to Def. 5. book V. of Euclid; and therefore this proposition is demonstrated by Propositions 16, B, 18, 17, E of the same book.

Otherwise algebraically.

Let $a : b :: c : d$, and therefore $ad = bc$.

Altern. $a : c :: b : d$
 Invert. $b : a :: d : c$
 Divid. $a - b : b :: c - d : d$
 Comp. $a + b : b :: c + d : d$
 Convert. $a : a - b :: c : c - d$

For since $ad = bc$, it is obvious, that in each of these cases the product of the extremes is equal to the product of the means; the quantities are therefore proportionals. (prop. 1.)

C H A P. III.

Of Equations in general, and of the Solution of simple Equations.

D E F I N I T I O N S.

- I. **A**N Equation may in general be defined, to be a proposition asserting the equality of two quantities; and is expressed by placing the sign $=$ between them.
- II. When a quantity stands alone upon one side of an equation, the quantities on the other side are said to be a *value* of it. Thus in the equation $x = b + y - d$, x stands alone on one side, and $b + y - d$ is a value of it.
- III. When an unknown quantity is made to stand alone on one side of an equation, and there are only known quantities

on the other; that equation is said to be *resolved*; and the value of the unknown quantity is called a *root* of the equation.

As the general relations of quantity which may be treated of in Algebra, are almost universally either that of equality, or such as may be reduced to that of equality, the doctrine of equations becomes one of the chief branches of the science.

The most common and useful application of Algebra is in the investigation of quantities that are unknown, from certain given relations to each other, and to such as are known, and hence it is called the *Analytical Art*. The equations employed for expressing these relations must therefore contain one or more unknown quantities, and the principal business of this art will be, the deducing equations, containing only one unknown quantity, and resolving them.

Def. 4. Equations containing only one unknown quantity and its powers, are divided into orders according to the highest power of the unknown quantity to be found in any of its terms.

If the highest power of the unknown quantity in any term be the $\left. \begin{array}{l} 1^{\text{st}}, \\ 2^{\text{d}}, \\ 3^{\text{d}}, \text{ \&c.} \end{array} \right\}$ The Equation is called $\left\{ \begin{array}{l} \text{Simple,} \\ \text{Quadratic,} \\ \text{Cubic, \&c.} \end{array} \right.$

But the exponents of the unknown quantity are supposed to be integers, and the equation is supposed to be cleared of fractions in which the unknown quantity or any of its powers enter the denominators. Thus $x + a = \frac{3x - b}{c}$ is a simple equation; $3x - \frac{5}{2x} = 12$, when cleared of the fraction by multiplying both sides by $2x$ becomes $6x^2 - 5 = 12x$ a quadratic. $x^3 - 2x^4 = x^6 - 20$ is an equation of the 6th order, &c.

The solution of the different orders of equations will be successively explained, the preliminary rules in the following section are useful in all orders, and are alone sufficient for the solution of simple equations.

S E C T. I. *Of simple Equations, and their Resolution.*

Simple equations are resolved by the four fundamental operations already explained; and the application of them to this purpose is contained in the following rules.

Rule I. *Any quantity may be transposed from one side of an equation to the other, by changing its sign.*

Thus, if $3 - 10 = 2x + 5$

Then,

Then, $3x - 2x = 10 + 5$ or $x = 15$

Thus also, $5x + b = a + 2x$

By transp. $3x = a - b$.

This rule is obvious from prob. 1. and 2. and by it, all the terms of the equation which involve the unknown quantity, may be brought to one side.

Cor. The signs of all the terms of an equation may be changed into the contrary signs, and it will continue to be true.

Rule II. *Any quantity by which the unknown quantity is multiplied, may be taken away, by dividing all the other quantities of the equation by it.*

Thus, if $ax = b$

$$x = \frac{b}{a}$$

Also, if $mx + nb = am$

$$x + \frac{nb}{m} = a$$

For if equal quantities are divided by the same quantity, the quotients are equal.

Rule III. *If a term of an equation is fractional, its denominator may be taken away, by multiplying all the other terms by it.*

Thus; if $\frac{x}{a} = b + c$ Also, if $a - \frac{b}{x} = c$

$$x = ab + ac \qquad ax - b = cx$$

And by transp. $ax - cx = b$

$$\text{And by div. } x = \frac{b}{a - c}$$

For if all the terms of the equation are multiplied by the same quantity, it remains a true proposition. Hence, if the unknown quantity is divided by any quantity, that quantity may be taken away.

Corollary to the three last rules.

If any quantity be found on both sides of the equation, with the same sign, it may be taken away from both. (Rule I.)

Also, if all the terms in the equation are multiplied or divided by the same quantity, it may be taken out of them all. (Rule II. and III.)

Ex.

Ex. If $3x + a = a + b$, then $3x = b$.

If $2ax + 3ab = ma + a^2$, then $2x + 3b = m + a$.

If $\frac{x}{3} - \frac{4}{3} = \frac{16}{3}$ then $x - 4 = 16$.

Examples of simple equations resolved by these rules.

I.

If $3x + 5 = x + 9$

R. 1. $2x = 4$

R. 2. $x = \frac{4}{2} = 2$

II.

If $5x - \frac{5x}{2} + 12 = \frac{4x}{3} + 26$

R. 1. $5x - \frac{5x}{2} - \frac{4x}{3} = 14$

R. 3. $30x - 15x - 8x = 84$

Or $7x = 84$

R. 2. $x = \frac{84}{7} = 12$

III.

If $\frac{5}{x} + \frac{9}{4} = 16$

R. 3. $\frac{20}{x} + 9 = 64$

R. 3. $20 + 9x = 64x$

R. 1. $20 = 55x$

R. 2. $x = \frac{20}{55} = \frac{4}{11}$

Sect. II. *Solution of Questions producing Simple Equations.*

From the resolution of equations we obtain the resolution of a variety of useful problems, both in pure mathematics and physics, and also in the practical arts founded upon these sciences. In this place, we consider the application of it to those questions where the quantities are expressed by numbers, and their magnitude alone is to be considered.

The rules given in this section for the solution of questions producing simple equations are applicable to questions which
pro.

produce equations of any order.

When an equation containing only one unknown quantity is deduced from the question by the following rules, it is sometimes called a *final Equation*. If it be simple, it may be resolved by the preceeding rules, but if it be of a superior order, it must be resolved by the rules afterwards to be explained. The examples in this chapter are so contrived that the final equation may be simple.

GENERAL RULE.

The unknown quantities in the question proposed must be expressed by letters, and the relations of the known and unknown quantities contained in it, or the conditions of it, as they are called, must be expressed by equations. These equations being resolved by the following rules, will give the answer of the question.

Thus if the sum of two numbers sought be } $x + y = 60$
60, that condition is expressed thus

If their difference must be 24, then - $x - y = 24$

If their product is 1640, then - - - $xy = 1640$

If their quotient must be 6, then - - $\frac{x}{y} = 6$

If their ratio is as 3 to 2, then $x : y :: 3 : 2$, } $2x = y$
and therefore

These are some of the obvious relations; many others occur in questions, which cannot be described in particular rules; and the algebraical expression of them must be left to the experience and judgment of the learner.

Case I. When there is only one unknown quantity to be found.

Rule. *An equation must be deduced from the question involving the unknown quantity (by the general rule) This equation being resolved by the rules of the last section, will give the answer.*

It is obvious, that when there is only one unknown quantity, there must be only one independent equation contained in the question; for any other would be unnecessary, and might be contradictory to the former.

EXAMPLE I.

To find a number, to which if there be added, a half, a third part, and a fourth part of itself, the sum will be 50.

Let

Let it be z , then half of it is $\frac{z}{2}$ a third of it $\frac{z}{3}$, &c.

$$\text{Therefore, } z + \frac{z}{2} + \frac{z}{3} + \frac{z}{4} = 50$$

$$24z + 12z + 8z + 6z = 1200$$

$$50z = 1200$$

$$z = 24.$$

If the operation be more complicated, it may be useful to register the several steps of it, as in the following

EXAMPLE II.

A trader allows 100l. per annum for the expences of his family, and augments yearly that part of his stock which is not so expended, by a third of it: at the end of three years his original stock was doubled: What had he at first?

Let his first stock be	1	z
Of which he spends the first year, 100 l. and there remains	2	$z - 100$
This remainder is increased by a third of itself	3	$z - 100 + \frac{z - 100}{3} = \frac{4z - 400}{3}$
The second year he spends 100 l. & there remains	4	$\frac{4z - 400}{3} - 100 = \frac{4z - 700}{3}$
He increases the remainder by one third of it	5	$\frac{4z - 700}{3} + \frac{4z - 700}{9} = \frac{16z - 2800}{9}$
The third year he spends 100 l. and there remains	6	$\frac{16z - 2800}{9} - 100 = \frac{16z - 3700}{9}$
He increases it by one third	7	$\frac{16z - 3700}{9} + \frac{16z - 3700}{27} = \frac{64z - 14800}{27}$
But at the end of the third year his stock is doubled; therefore	8	$\frac{64z - 14800}{27} = 2z$
By R. 3.	9	$64z - 14800 = 54z$
By R. 1.	10	$10z = 14800$
By R. 2.	11	$z = 1480$

Therefore his stock was 1480 l. which being tried answers the conditions of the question.

Cafe II. When there are two unknown quantities.

Rule: *Two independent equations must be derived from the question, involving the two unknown quantities. A value of one of the unknown quantities must be derived from each of the equations; and these two values being put equal to each other, a new equation will arise, involving only one unknown quantity, and may therefore be resolved by the preceding rule.*

Two equations must be deduced from the question; for from one including two unknown quantities, no solution can be obtained; more than two would be unnecessary or contradictory.

EXAMPLE III.

Two persons, A and B, were talking of their ages; jays A to B, Seven years ago I was just three times as old as you were, and seven years hence I shall be just twice as old as you will be: I demand their present ages.

Let the ages of A and B be respectively.

Seven years ago they were

Seven years hence they will be

Therefore by Quest. and 2.

Also by Quest. and 3.

By 4. and transp.

By 5. and transp.

By 6 and 7.

Transp. and 8.

By 9 and 6, or 7.

The ages of A and B then are 49 and 21, which answer the conditions.

$$1 \mid x \text{ and } y$$

$$2 \mid x - 7 \text{ and } y - 7$$

$$3 \mid x + 7 \text{ and } y + 7$$

$$4 \mid x - 7 = 3 \times y - 7 = 3y - 21$$

$$5 \mid x + 7 = 2 \times y + 7 = 2y + 14$$

$$6 \mid x = 3y - 14$$

$$7 \mid x = 2y + 7$$

$$8 \mid 3y - 14 = 2y + 7$$

$$9 \mid y = 21$$

$$10 \mid x = 49$$

The operation might have been a little shortened by subtracting the 4th from 5th, and thus $14 = -y + 35$ and hence $y = 21$. therefore (by 6th) $x = (3y - 14) = 49$.

EXAMPLE IV.

A gentleman distributing money among some poor people, found he wanted 10 s. to be able to give 5 s. to each; therefore he gives each 4s. only, and finds he has 5s. left.—To find the number of shillings and poor people?

D

Though

Though there are two unknown quantities, the question may be resolved by only one letter.

Let the number of shillings be

The number of poor will be

The number of poor is also

By 2. and 3.

Mult.

Transp.

$$\begin{array}{r|l}
 1 & z \\
 2 & z + 10 \\
 & \hline
 & 5 \\
 3 & z - 5 \\
 & \hline
 & 4 \\
 4 & z - 5 = z + 10 \\
 & \hline
 & 4 \qquad \qquad 5 \\
 5 & 5z - 25 = 4z + 40 \\
 6 & z = 65
 \end{array}$$

The number of shillings is 65, and the number of poor is $\left(\frac{z-5}{4} =\right) 15$, which answer the conditions.

EXAMPLE V.

A courier sets out from a certain place, and travels at the rate of 7 miles in 5 hours; and 8 hours after, another sets out from the same place, and travels the same road, at the rate of 5 miles in 3 hours: I demand how long and how far the first must travel, before he is overtaken by the second?

Let the number of hours	}	1	y	
which the first travelled be		2	$y - 8$	
Then the second travelled				
The first travelled seven	}	3	$(5 : 7 :: y :) \frac{7y}{5}$ miles	
miles in 5 hours, and				
therefore in y hours				
In like manner the second	}	4	$(3 : 5 :: y - 8 :) \frac{5y - 40}{3}$ miles	
travelled in $y - 8$ hours				
But they both travelled the	}	5	$\frac{5y - 40}{3} = \frac{7y}{5}$	
same number of miles;				
therefore by 3 and 4				
Mult. - - - - -	6	$25y - 200 = 21y$		
Transp. - - - - -	7	$4y = 200$		
Divid. - - - - -	8	$y = 50.$		

The first then travelled 50 hours, the second ($y - 8 =$) 42 hours.

The miles travelled by each $\left(\frac{7y}{5} = \frac{5y - 40}{3} =\right) 70.$

Case III. When there are three or more unknown quantities.

Rule

Rule. When there are three unknown quantities, there must be three independent equations arising from the question; and from each of these a value of one of the unknown quantities must be obtained. By comparing these three values, two equations will arise involving only two unknown quantities, which may therefore be resolved by the rule for case 2.

In like manner may the rule be extended to such questions as contain four or more unknown quantities; and hence it may be inferred, *That when just as many independent equations may be derived from a question, as there are unknown quantities in it, these quantities may be found by the preceeding rules.*

EXAMPLE VI.

To find three numbers, so that the first, with half the other two, the second with one third of the other two, and the third with one fourth of the other two, may each be equal to 34.

Let the numbers be x, y, z , and the equations are

	1	$x + \frac{y+z}{2} = 34$
	2	$y + \frac{x+z}{3} = 34$
	3	$z + \frac{x+y}{4} = 34$
From the 1st, -	4	$x = \frac{68 - y - z}{2}$
From the 2d, -	5	$x = 102 - 3y - z$
From the 3d, -	6	$x = 136 - 4z - y$
From 4th and 5th -	7	$\frac{68 - y - z}{2} = 102 - 3y - z$
7th reduced -	8	$y = \frac{136 - z}{5}$
5th and 6th reduced -	9	$y = \frac{3z - 34}{2}$
8 and 9 -	10	$\frac{3z - 34}{2} = \frac{136 - z}{5}$
10th reduced -	11	$15z - 170 = 272 - 2z$
	12	$17z = 442 \text{ or } z = 26$
By 8 and 5 -	13	$y = 22 \text{ and } x = 10$

E X.

EXAMPLE VII.

There is a certain number consisting of three places, whose digits are in arithmetical proportion; if this number be divided by the sum of its digits, the quotient will be 48; and lastly, if from the number be subtracted 198, the digits will be inverted.

Let the 3 digits be	1	x, y, z
Then the number is	2	$100x + 10y + z$
If the digits be	3	$100z + 10y + x$
inverted, it is		
The digits are	4	$x + z = 2y$
in ar. prop.		
therefore		
By question	5	$\frac{100x + 10y + z}{x + y + z} = 48$
By question	6	$100x + 10y + z - 198 = 100z + 10y + x$
From 6 and transf.	7	$99x = 99z + 198$
Divid. by 99	8	$x = z + 2$
From 4	9	$x = 2y - z$
8 and 9	10	$2y - z = z + 2$
Transp.	11	$y = z + 1$
Mult. 5.	12	$100x + 10y + z = 48x + 48y + 48z$
Transp.	13	$52x = 38y + 47z$
8 and 11 substit.	14	$52z + 104 = 38z + 38 + 47z$
for x and y	15	$33z = 66$
Transf.	16	$\left\{ \begin{array}{l} z = 2 \\ y = (z + 1) 3 \\ x = (z + 2) 4. \end{array} \right.$
Divid.		

The number then is 432, which succeeds upon trial.

Corollary to the preceeding rules.

It appears, that in every question, there must be as many independent equations as unknown quantities; if there are not, then the question is called *indeterminate*, because it may admit of an infinite number of answers; since the equations wanting, may be assumed at pleasure. There may be other circumstances however, to limit the answers to one, or a precise number, and which, at the same time, cannot be directly expressed by equations. Such are these; that the numbers must be integers, squares, cubes,

cubes, and many others. The solution of such problems, which are also called Diophantine, shall be considered afterwards.

SCHOLIUM.

On many occasions, by particular contrivances, the operations by the preceeding rules may be much abridged. This however, must be left to the skill and practice of the learner. A few examples are the following.

1. It is often easy to employ fewer letters than there are unknown quantities, by expressing some of them from a simple relation to others. Thus, the solution becomes more easy and elegant. (See Ex. 4, 5.)

2. Sometimes it is convenient to express by letters, not the unknown quantities themselves, but some other quantities connected with them, as their sum, difference, &c. from which they may be easily derived. (See Ex. 1. of chap. 5.)

3. In the operation also, circumstances will suggest a more easy road than that pointed out by the general rules. Two of the original equations may be added together, or may be subtracted; various substitutions may be made, &c. which will render the solution much less complicated. (See Ex. 3. and 7.)

CHAP. III. PART II.

General Solution of Problems.

IN the solutions of the questions in the preceeding part, the given quantities (being numbers) disappear in the last conclusion, so that no general rules for like cases can be deduced from them. But if letters are made to denote the known quantities, as well as the unknown, a general solution may be obtained, because, during the whole course of the operation, they retain their original form. Hence also the connection of the quantities will appear in such a manner as to discover the necessary limitations of the data, when there are any, which is essential to the perfect solution of a problem. From this method too, it is easy to derive a synthetical demonstration of the solution.

When letters, or any other such symbols, are employed to express all the quantities, the algebra is called *specious* or *literal*.

E X.

EXAMPLE VIII.

To find two numbers, of which the sum and difference are given.

Let s be the given sum, and d the given difference. Also, let x and y be the two numbers sought.

$$\begin{aligned} x + y &= s \\ x - y &= d \\ \text{Whence } \begin{cases} x = s - y \\ x = d + y \end{cases} \\ d + y &= s - y \\ 2y &= s - d \\ y &= \frac{s - d}{2} \end{aligned}$$

$$\text{And } x = \frac{s + d}{2}$$

Thus let the given sum be 100, and the difference 24

$$\text{Then } x = \left(\frac{s + d}{2} = \frac{124}{2} = \right) 62 \text{ and } y = \left(\frac{s - d}{2} = \frac{76}{2} = \right) 38.$$

In the same manner may the canon be applied to any other values of s and d . By reverfing the fteps in the operation, it is eafy to fhew, that if $x = \frac{s + d}{2}$ and $y = \frac{s - d}{2}$ the fum of x and y muft be s , and their difference d .

EXAMPLE IX.

If A and B together can perform a piece of work in the time a, A and C together in the time b, and B and C together in the time d, in what time will each of them perform it alone?

Let the time in which A performs it be x , B in y , and C in z , then as the work is the fame in all cafes it may be represented by unity.

By

By the question

$$\left. \begin{array}{l}
 1 \ (x : 1 :: a :) \frac{a}{x} = \\
 2 \ (y : 1 :: a :) \frac{a}{y} = \\
 3 \ (x : 1 :: b :) \frac{b}{x} = \\
 4 \ (z : 1 :: b :) \frac{b}{z} = \\
 5 \ (y : 1 :: c :) \frac{c}{y} = \\
 6 \ (z : 1 :: c :) \frac{c}{z} =
 \end{array} \right\} \begin{array}{l} \text{to the work performed by} \\ A \text{ in } a \text{ days} \\ B \text{ in } a \text{ days} \\ A \text{ in } b \text{ days} \\ C \text{ in } b \text{ days} \\ B \text{ in } c \text{ days} \\ C \text{ in } c \text{ days} \end{array}$$

$$7 \ \frac{a}{x} + \frac{a}{y} = 1 \text{ and } ay + ax = xy$$

$$8 \ \frac{b}{x} + \frac{b}{z} = 1 \text{ and } bz + bx = xz$$

$$9 \ \frac{c}{y} + \frac{c}{z} = 1 \text{ and } cz + cy = zy$$

Mult. 7th by $\frac{bc}{xy}$

$$10 \ \frac{abc}{x} + \frac{abc}{y} = bc$$

Mult. 8th by $\frac{ac}{xz}$

$$11 \ \frac{abc}{x} + \frac{abc}{z} = ac$$

Mult. 9th by $\frac{ab}{zy}$

$$12 \ \frac{abc}{y} + \frac{abc}{z} = ab$$

Add 10th, 11th, 12th

$$13 \ \frac{2abc}{x} + \frac{2abc}{y} + \frac{2abc}{z} = bc + ac + ab$$

From 13th subtr. twice 10th

$$14 \ \frac{2abc}{z} = ac + ab - bc \ \& \ z = \frac{2abc}{ac + ab - bc}$$

From 13th subtr. twice 11th

$$15 \ \frac{2abc}{y} = bc + ab - ac \ \& \ y = \frac{2abc}{bc + ab - ac}$$

From 13th subtr. twice 12th

$$16 \ \frac{2abc}{x} = bc + ac - ab \ \& \ x = \frac{2abc}{bc + ac - ab}$$

Example in numbers let $a = 8$ days $b = 9$ days and $c = 10$, then $x = 14 \frac{34}{49}$, $y = 17 \frac{23}{41}$, and $z = 23 \frac{7}{31}$. It appears likewise that

a, b, c , must be such, that the product of any two of them must be less than the sum of these two multiplied by the third. This is necessary to give positive values of x, y , and z , which alone can take place in this question.

E X.

EXAMPLE X.

In question 5th, let the first courier travel p miles in q hours; the second r miles in s hours; let the interval between their setting out be a .

Then by working as formerly.

$$x = \frac{qra}{qr - ps}.$$

If particular values be inserted for these letters, a particular solution will be obtained for that case. Let them denote the numbers in Example 5.

$$\text{Then } x = \left(\frac{qra}{qr - ps} = \frac{5 \times 5 \times 8}{5 \times 5 - 7 \times 3} = \frac{200}{4} = \right) 50.$$

Here it is obvious, that qr must be greater than ps , else the problem is impossible; for then the value of x would either be infinite or negative. This limitation appears also from the nature of the question, as the second courier must travel at a greater rate than the first, in order to overtake him.

C H A P. IV.

Of Involution and Evolution.

IN order to resolve equations of the higher orders, it is necessary to premise the rules of INVOLUTION and EVOLUTION.

L E M M A.

The reciprocals of the powers of a quantity may be expressed by that quantity, with negative exponents of the same denomination. That is, the series $a, 1, \frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^m}$, &c. may be expressed by $a^1, a^0, a^{-1}, a^{-2}, a^{-3}, a^{-m}$, &c.

For the rule for dividing the powers of the same root was to subtract the exponents; if then the index of the divisor be greater than that of the dividend, the index of the quotient must be negative.

$$\text{Thus, } \frac{a^2}{a^3} = a^{2-3} = a^{-1}. \text{ Also, } \frac{a^2}{a^3} = \frac{1}{a^1}.$$

$$\frac{a^m}{a^m} = a^{m-m} = a^0. \text{ And, } \frac{a^m}{a^m} = 1. \text{ and so on of others.}$$

Cor. 1. Hence any quantity which multiplies either the numerator or denominator of a fraction, may be transposed from the one to the other, by changing the sign of its index.

$$\text{Thus, } \frac{x}{y^2} = xy^{-2}. \text{ And } \frac{a^2x}{y^3} = \frac{a^2}{y^3x^{-1}}, \text{ \&c.}$$

Cor. 2. From this notation, it is evident that these *negative powers*, as they are called, are multiplied by adding, and divided by subtracting their exponents.

$$\text{Thus, } a^{-2} \times a^{-3} = a^{-5}.$$

$$\text{Or, } \frac{1}{a^2} \times \frac{1}{a^3} = \frac{1}{a^5} = a^{-5}.$$

$$\frac{a^{-1}}{a^{-3}} = a^2 \text{ Or, } \left(\frac{1}{a^3} \right) \frac{1}{a} \left(\frac{a^3}{a} \right) = a^2.$$

I. Of Involution.

To find any power of any quantity, is the business of Involution.

Case I. When the quantity is simple.

Rule. *Multiply the exponents of the letters by the index of the power required, and raise the coefficient to the same power.*

$$\text{Thus, the 2d power of } a, \text{ is } a^1 \times 2 = a^2$$

$$\text{The 3d power of } 2a^2, \text{ is } 8a^2 \times 3 = 8a^6$$

$$\text{The 3d power of } 3ab^3, \text{ is } 27a^1 \times 3b^3 \times 3 = 27a^3b^9.$$

For the multiplication would be performed by the continued addition of the exponents; and this multiplication of them is equivalent. The same rule holds also when the signs of the exponents are negative.

Rule for the Signs.

If the sign of the given quantity is +, all its powers must be positive. If the sign is -, then all its powers whose exponents are even numbers, are positive; and all its powers whose exponents are odd numbers, are negative.

This is obvious from the rule for the signs in multiplication. It implies the most extensive use of the signs + and -.

Case II. When the quantity is compound.

E

Rule,

Rule. The powers must be found by a continual multiplication of it by itself.

Thus, the square of $x + \frac{a}{2}$ is found by multiplying it into itself. The product is $x^2 + ax + \frac{a^2}{4}$. The cube of $x + \frac{a}{2}$ is got by multiplying the square already found by the root, &c.

Fractions are raised to any power, by raising both numerator and denominator to that power, as is evident from the rule for multiplying fractions, in chap. 1. p. 2.

The involution of compound quantities is rendered much easier by the binomial theorem; for which, see chap. VI.

Note. The square of a binomial consists of the squares of the two parts, and twice the product of the two parts.

II. Of Evolution.

Evolution is the reverse of involution, and by it powers are resolved into their roots.

Def. The root of any quantity is expressed by placing before it $\sqrt{\quad}$ (called a *radical sign*) with a small figure above it, denoting the denomination of that root.

Thus, the square root of a , is $\sqrt[2]{a}$ or $\sqrt{a}$

The cube root of bc , is $\sqrt[3]{bc}$

The 4th root of $a^2b - x^3$, is $\sqrt[4]{a^2b - x^3}$

The m th root of $c^2 - dx$, is $\sqrt[m]{c^2 - dx}$

General Rule for the Signs.

1. The root of any positive power may be either positive or negative, if it is denominated by an even number; if the root is denominated by an odd number, it is positive only.
2. If the power is negative, the root also is negative, when it is denominated by an odd number.
3. If the power is negative, and the denomination of the root even, then no root can be assigned.

This rule is easily deduced from that given in involution. In the last case, though no root can be assigned, yet sometimes it

is convenient to set the radical sign before the negative quantity, and then it is called an *impossible* or *imaginary* root.

The radical sign may be employed to express any root of any quantity whatever, but sometimes the root may be accurately found by the following rules, and when it cannot, it may often be more conveniently expressed by the methods now to be explained.

Case I. When the quantity is simple.

Rule. *Divide the exponents of the letters by the index of the root required, and prefix the root of the coefficient.*

1. The exponents of the letters may be multiples of the index of the root, and the root of the coefficient may be extracted.

Thus the square root of $a^4 = a^{\frac{4}{2}} = a^2$

$$\sqrt[3]{27a^6} = 3a^{\frac{6}{3}} = 3a^2$$

$$\sqrt[4]{a^4b^{12}} = a^{\frac{4}{4}}b^{\frac{12}{4}} = ab^3.$$

2. The exponents of the letters may not be multiples of the index of the root, and then they become fractions: and when the root of the coefficient cannot be extracted, it may also be expressed by a fractional exponent, its original index being understood to be 1.

$$\text{Thus, } \sqrt{16a^3b^2} = 4a^{\frac{3}{2}}b$$

$$\sqrt[3]{7ax^3} = 7^{\frac{1}{3}}a^{\frac{1}{3}}x = \sqrt[3]{7} \times a^{\frac{1}{3}}x.$$

As evolution is the reverse of involution, the reason of the rule is evident.

The root of any fraction is found by extracting that root out of both numerator and denominator.

Case II. When the quantity is compound.

1. To extract the square root.

R U L E.

1. *The given quantity is to be ranged according to the powers of the letters, as in division.*

Thus, in the example $a^2 + 2ab + b^2$, the quantities are ranged in this manner.

2. *The square root is to be extracted out of the first term, (by pre-*
ceed-

ceding rules) which gives the first part of the root sought. Subtract its square from the given quantity, and divide the first term of the remainder by double the part already found, and the quotient is the second term of the root.

Thus, in this example, the remainder is $2ab + b^2$; and $2ab$ being divided by $2a$, the double of the part found, gives $+b$ for the second part of the root.

3. Add this second part to double of the first, and multiply their sum by the second part: subtract the product from the last remainder, and if nothing remain, the square root is obtained. But if there is a remainder, it must be divided by the double of the parts already found, and the quotient would give the third part of the root: and so on.

In the last example, it is obvious, that $a + b$ is the square root sought.

The entire operation is as follows.

$$\begin{array}{r}
 a^2 + 2ab + b^2 \quad (a + b \\
 \underline{a^2} \\
 2a + b \quad + 2ab + b^2 \\
 \times b \quad \underline{2ab + b^2} \\
 * \\
 x^4 - ax^3 + \frac{a^2}{4} \left(x^2 - \frac{a}{2} \right. \\
 \underline{x^4} \\
 2x^2 - \frac{a}{2} \quad - ax^3 + \frac{a^2}{4} \\
 \times - \frac{a}{2} \quad - ax^3 + \frac{a^2}{4} \\
 \underline{\hspace{1.5cm}} \\
 *
 \end{array}$$

The reason of this rule appears from the composition of a square.

2. To extract any other root.

Rule. Range the quantity according to the dimensions of its letters, and extract the said root out of the first term, and that shall be the first member of the root required. Then raise this root to a dimension lower by unit, than the number that denominates the root required, and multiply the power that arises, by that number itself: divide the second term of the given quantity by the product, and the quotient shall give the second member of the root required. In like manner are the other parts to be found, by considering those already got as making one term.

Thus

Thus the fifth root of
 $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 (a + b)$
 a^5

$$5a^4 \quad) \quad 5a^4b$$

And $a + b$ raised to the 5th power is the given quantity, and therefore it is the root sought.

In evolution it will often happen, that the operation will not terminate, and the root will be expressed by a series.

Thus the square root of $a^2 + x^2$ becomes a series.

$$a^2 + x^2 \left(a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5}, \&c. \right)$$

$$a^2$$

$$2a + \frac{x^2}{2a} \quad) \quad * + x^2$$

$$\times \frac{x^2}{2a} = x^2 + \frac{x^4}{4a^2}$$

$$2a + \frac{x^2}{a} - \frac{x^4}{8a^3} \quad) \quad * - \frac{x^4}{4a^2}$$

$$\times -\frac{x^4}{8a^3} = -\frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6}$$

$$* + \frac{x^6}{8a^4} - \frac{x^8}{64a^6}$$

&c.

The extraction of roots by series is much facilitated by the binomial theorem. By similar rules are the roots of numbers to be extracted.

III. Of Surds.

Def. Quantities with fractional exponents are called *Surds* or *Imperfect Powers*.

Such quantities are also called *irrational*, in opposition to others with integral exponents, which are called *rational*.

Surds may be expressed either by the fractional exponents, or by the radical sign, the denominator of the fraction being its index; and hence the orders of surds are denominated from this index.

In

In the following operations however, it is generally convenient to use the notation by the fractional exponents.

$$a^{\frac{1}{3}} = \sqrt[3]{a} \quad \sqrt[3]{4ab^3} = 2ba^{\frac{1}{3}} \sqrt[3]{a^3b^3} = a^{\frac{3}{3}}b^{\frac{1}{3}}.$$

The operations concerning surds depend on the following principle. *If the numerator and denominator of a fractional exponent be both multiplied or both divided by the same quantity, the value of the power is the same.* Thus $a^{\frac{m}{n}} = a^{\frac{mc}{nc}}$; for let $a^{\frac{m}{n}} = b$ then $a^m = b^n$ and $a^{mc} = b^{nc}$ and extracting the root nc , $a^{\frac{mc}{nc}} = b^{\frac{nc}{nc}} = b = a^{\frac{m}{n}}$.

Lem. A rational quantity may be put into the form of a surd, by reducing its index to the form of a fraction of the same value.

$$\text{Thus } a = a^{\frac{3}{2}} = \sqrt{a^3}$$

$$a^3b = a^{\frac{6}{3}}b^{\frac{3}{3}} = \sqrt[3]{a^6b^3}$$

Prob. I. To reduce surds of different denominations to others of the same value, and of the same denomination.

Rule. *Reduce the fractional exponents to others of the same value and having the same common denominator.*

$$\text{Ex. } \sqrt{a}, \sqrt[3]{b^2} \text{ or } a^{\frac{1}{2}}, b^{\frac{2}{3}}$$

$$\text{but } a^{\frac{1}{2}} = a^{\frac{3}{6}} \text{ and } b^{\frac{2}{3}} = b^{\frac{4}{6}}.$$

therefore $\sqrt{a}$, and $\sqrt[3]{b^2}$ are respectively equal to $\sqrt[6]{a^3}$ and $\sqrt[6]{b^4}$.

Prob. II. To multiply and divide surds.

1. *When they are surds of the same rational quantity, add and subtract their exponents.*

$$\text{Thus, } a^{\frac{1}{3}} \times a^{\frac{2}{3}} = a^{\frac{1}{3} + \frac{2}{3}} = a^{\frac{3}{3}} = \sqrt[3]{a^3}$$

$$\frac{\sqrt[3]{a^3b^2}}{\sqrt[3]{a^2b}} = \frac{a^{\frac{3}{3}}b^{\frac{2}{3}}}{a^{\frac{2}{3}}b^{\frac{1}{3}}} = a^{\frac{3}{3} - \frac{2}{3}}b^{\frac{2}{3} - \frac{1}{3}} = a^{\frac{1}{3}}b^{\frac{1}{3}} = \sqrt[3]{ab^1}.$$

2. *If they are surds of different rational quantities, let them be brought to others of the same denomination, if already they are not, by prob. I. Then by multiplying or dividing these rational*
quan-

quantities, their product or quotient may be set under the common radical sign.

$$\text{Thus } \sqrt{a} \times \sqrt{b} = a^{\frac{1}{2}} b^{\frac{1}{2}} = \sqrt{ab}.$$

$${}^m\sqrt{a} \times {}^n\sqrt{b} = a^{\frac{1}{m}} b^{\frac{1}{n}} = {}^{mn}\sqrt{a^n b^m}.$$

If the surds have any rational coefficients, their product or quotient must be prefixed. Thus $a\sqrt{m} \times b\sqrt{n} = ab\sqrt{mn}$.

Cor. If a rational coefficient be prefixed to a radical sign, it may be reduced to the form of a surd by the lemma, and multiplied by this problem; and conversely, if the quantity under the radical sign be divisible by a perfect power of the same denomination, it may be taken out, and its root prefixed as a coefficient.

$$a\sqrt{b} = \sqrt{a^2 b}; \quad 2 \times^3 \sqrt{a} = {}^3\sqrt{8a}.$$

$$\text{Conv. } \sqrt{a^2 b} = ab\sqrt{b}; \quad \sqrt{4a^2 - 8a^2 b} = 2a\sqrt{1 - 2b}.$$

Even when the quantity under the radical sign is not divisible by a perfect power it may be useful sometimes to divide surds into their component factors by reversing the operation of this problem.

$$\begin{aligned} \text{Thus } \sqrt{ab} &= \sqrt{a} \times \sqrt{b}, \quad {}^3\sqrt{a^2 b - bx^2} = {}^3\sqrt{ba - bx} \times {}^3\sqrt{a+x} \\ &= b^{\frac{1}{3}} \times {}^3\sqrt{a-x} \times {}^3\sqrt{a+x}. \end{aligned}$$

Prob. III. To involve or evolve surds.

This is performed by the same rules as in other quantities, by multiplying or dividing their exponents by the index of the power, or root required.

SCHOLIUM.

If a member of an equation be a surd root, then the equation may be freed from any surd by bringing that member first to stand alone upon one side of the equation, and then taking away the radical sign from it, and raising the other side to the power denominated by the surd.

This operation becomes a necessary step towards the solution of an equation, when any of the unknown quantities are under the radical sign.

$$\text{Example, If } 3\sqrt{x^2 - a^2} + 2y = a + y$$

$$\text{Then } 3\sqrt{x^2 - a^2} = a - y$$

$$\text{and } 9 \times x^2 - a^2 = a^2 - 2ay + y^2$$

If

If the unknown quantity be found only under the radical sign and only of the first dimension, the equation will become simple and may be resolved by the preceeding rules.

$$\text{Thus if } \sqrt[3]{4x + 16} + 5 = 9$$

$$\text{Then, } \sqrt[3]{4x + 16} = 4$$

$$\text{And } 4x + 16 = 64$$

$$4x = 48$$

$$\text{And, } x = 12$$

$$\text{If } \sqrt[m]{a^2x - b^2x} = a$$

$$\text{Then, } a^2x - b^2x = a^m$$

$$x = \frac{a^m}{a^2 - b^2}$$

If the unknown quantity in a final equation has fractional exponents, by means of the preceeding rules, a new equation may be substituted in which the exponents of the unknown quantity are integers.

Thus, if $x^{\frac{1}{2}} + 3x^{\frac{2}{3}} = 10$, by reducing the surds to the same denomination it becomes $x^{\frac{3}{6}} + 3x^{\frac{4}{6}} = 10$; and if $z = x^{\frac{1}{6}}$ then $z^3 + 3z^4 = 10$, and if this equation be resolved from a value of z a value of x may be got by the rules of the next chapter. Thus also if $x + 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} = 100$. If $x^{\frac{1}{6}} = z$, this equation becomes $z^6 + 2z^3 - 3z^2 = 100$. In general if $x^{\frac{p}{q}} + x^{\frac{m}{n}} = a$, by reducing the surds to the same denomination $x^{\frac{pn}{qn}} + x^{\frac{qm}{qn}} = a$, and if $x^{\frac{1}{qn}} = z$ then the equation is $zp^n + zq^m = a$ in which the exponents of z are integers, and z being found x is to be found from the equation $x^{\frac{1}{qn}} = z$, viz. $x = z^{qn}$.

C H A P. V.

Equations were divided into orders according to the highest index of the unknown quantity in any term. (Chap. III.)

Equations are either *pure* or *affected*.

Def. 1. A *pure Equation*, is that in which only one power of the unknown quantity is found

2. An

2. An *affected Equation*, is that in which different powers of the unknown quantity are found in the several terms. Thus, $a^2 + ax^2 = b^3$, $ax^2 - b^2 = m^2 + x^2$ are pure equations. And $x^2 - ax = b^2$, $x^3 + x^2 = 17$, are affected.

I. Solution of pure Equations.

Rule. Make the power of the unknown quantity, to stand alone by the rules formerly given, and then extract the root of the same denomination out of both sides, which will give the value of the unknown quantity.

E X A M P L E S.

$$\text{If } a^2 + ax^2 = b^3$$

$$ax^2 = b^3 - a^2$$

$$x^2 = \frac{b^3 - a^2}{a}$$

$$x = \sqrt{\frac{b^3 - a^2}{a}}$$

$$ax^m - b = x^m - c$$

$$ax^m - x^m = b - c$$

$$x^m = \frac{b - c}{a - 1}$$

$$x = \sqrt[m]{\frac{b - c}{a - 1}}$$

The index of the power may also be fractional; as in the last example m may be any number whatever. Let $m = \frac{1}{2}$, then as before,

$$x^m = x^{\frac{1}{2}} = \frac{b - c}{a - 1}$$

$$\text{And } x = \left(\frac{b - c}{a - 1} \right)^2 = \frac{b^2 - 2bc + c^2}{a^2 - 2a + 1}$$

Sometimes different powers of the unknown quantity are found in the equation, yet the several terms may form on one side a perfect power, of which the root being extracted, the equation will become simple.

Thus, if $x^3 - 12x^2 + 48x = 98$. It is easy to observe that $x^3 - 12x^2 + 48x - 64 = 34$; forming a complete cube, of which the root being extracted, $x - 4 = \sqrt[3]{34}$. And $x = 4 + \sqrt[3]{34}$.

E X A M P L E I.

To find four continued proportionals, of which the sum of the extremes is 56, and the sum of the means 24.

To resolve the question in general terms, let the sum of the extremes be a , the sum of the means b , and let the difference of the

extremes be called z , and the difference of the means y . Then by Ex. 8. Chap. 2.

The proportionals are	1	$\frac{a+z}{2} : \frac{b+y}{2} : \frac{b-y}{2} : \frac{a-z}{2}$
Mult. by 2 and still	2	$a+z : b+y : b-y : a-z$
From the three first	3	$ab - ay + bz - zy = b^2 + 2by + y^2$
From the three last	4	$1b + ay - bz - zy = b^2 - 2by + y^2$
3d added to 4th	5	$2ab - 2zy = 2b^2 + 2y^2$
4th subtr. from 3d	6	$2bz - 2ay = 4by$
6th reduced	7	$z = \frac{2by + ay}{b}$
7th subst. for z in 5th	8	$2ab - 2 \times \frac{2by + ay}{b} = 2b^2 + 2y^2$
Transp. and divide 8th by $\frac{2}{b}$	9	$ab^2 - b^3 = 3by^2 + ay^2$
In numbers	10	$\frac{ab^2 - b^3}{3b + a} = y^2$ and $y = \sqrt{\frac{ab^2 - b^3}{3b + a}}$
	11	$y = \sqrt{\frac{ab^2 - b^3}{3b + a}} = b \sqrt{\frac{a-b}{3b + a}} = 24 \sqrt{\frac{32}{128}} = 12$
	12	$z = \frac{2b + a}{b} \times y = 52$

Hence the four proportionals are 54, 18, 6, 2; and it appears that b must not be greater than a , otherwise the root becomes impossible, and the problem would also be impossible; which limitation might be deduced also from prop. 25. 5th of Euclid.

II. Solution of adfected Quadratic Equations.

An adfected quadratic equation, (commonly called a quadratic) involves the unknown quantity itself, and also its square: It may be resolved by the following

R U L E.

1. *Transpose all the terms involving the unknown quantity, to one side, and the known terms to the other; and so that the term containing the square of the unknown quantity may be positive.*

2. *If the square of the unknown quantity is multiplied by any*
co-

coefficient, all the terms of the equation are to be divided by it, so that the coefficient of the square of the unknown quantity may be 1.

3. Add to both sides the square of half the coefficient of the unknown quantity itself, and the side of the equation involving the unknown quantity, will be a complete square.
4. Extract the square root from both sides of the equation, by which it becomes simple, and by transposing the above mentioned half coefficient, a value of the unknown quantity is obtained in known terms, and therefore the equation is resolved.

The reason of this rule is manifest from the composition of the square of a binomial, for it consists of the squares of the two parts, and twice the product of the two parts. (Chap. IV.)

The different forms of quadratic equations being reduced by the first and second parts of the rule, are these ;

$$1. \quad x^2 + ax = b^2$$

$$2. \quad x^2 - ax = b^2$$

$$3. \quad x^2 - ax = -b^2$$

$$\text{Case 1.} \quad x^2 + ax = b^2$$

$$\begin{aligned} x^2 + ax + \frac{a^2}{4} &= b^2 + \frac{a^2}{4} \\ x + \frac{a}{2} &= \pm \sqrt{b^2 + \frac{a^2}{4}} \\ x &= \pm \sqrt{b^2 + \frac{a^2}{4}} - \frac{a}{2} \end{aligned}$$

$$\text{Case 2.} \quad x^2 - ax = b^2$$

$$\begin{aligned} x^2 - ax + \frac{a^2}{4} &= b^2 + \frac{a^2}{4} \\ x - \frac{a}{2} &= \pm \sqrt{b^2 + \frac{a^2}{4}} \\ x &= \frac{a}{2} \pm \sqrt{b^2 + \frac{a^2}{4}} \end{aligned}$$

$$\text{Case 3.} \quad x^2 - ax = -b^2$$

$$\begin{aligned} x^2 - ax + \frac{a^2}{4} &= \frac{a^2}{4} - b^2 \\ x - \frac{a}{2} &= \pm \sqrt{\frac{a^2}{4} - b^2} \\ x &= \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b^2} \end{aligned}$$

Of these cases it may be observed ;

That

That if the square root of a positive quantity may be either positive or negative, according to the most extensive use of the signs every quadratic, equation will have two roots, except such of the third form, whose roots become impossible.

2. It is obvious, that in the two first forms one of the roots must be positive, and the other negative.

3. In the third form, if $\frac{a^2}{4}$ or the square of half the coefficient of the unknown quantity be greater than b^2 , the known quantity; the two roots will be positive. If $\frac{a^2}{4}$ be equal to b^2 , the two roots then become equal.

But if in this third case, $\frac{a^2}{4}$ is less than b^2 , the quantity under the radical sign becomes negative, and the two roots are therefore impossible. This may be easily shewn to arise from an impossible supposition in the original equation.

4. If the equation however, expresses the relation of magnitudes abstractly considered, where a contrariety cannot be supposed to take place; the negative roots cannot be of use, or rather there are no such roots; for then a negative quantity by itself is unintelligible, and therefore the square root of a positive quantity must be positive only. Hence, in the two first cases, there will be only one root; but in the third, there will be two. For in this third case, $x^2 - ax = -b^2$, or, $ax - x^2 = b^2$, it is obvious, that x may be either greater or less than $\frac{1}{2}a$, and yet $ax - x^2$ be equal to a given positive quantity b^2 : therefore the square root of $x^2 - ax + \frac{1}{4}a^2$ may be either $x - \frac{1}{2}a$ or, $\frac{1}{2}a - x$, and both these quantities also positive.

Let then $x - \frac{a}{2} = \sqrt{\frac{a^2}{4} - b^2}$ and $x = \frac{a}{2} + \sqrt{\frac{a^2}{4} - b^2}$. Also

let $\frac{a}{2} - x = \sqrt{\frac{a^2}{4} - b^2}$ or $x = \frac{a}{2} - \sqrt{\frac{a^2}{4} - b^2}$. and these are

the same two positive roots as were obtained by the general rule.

The general rule is usually employed even in questions where negative numbers cannot take place, and then the negative roots of the two first forms are neglected. Sometimes even, only one of the positive roots of the third case can be used, and the other may be excluded by a particular condition in the question. When an impossible root arises in the solution of a question, and if it be resolved in general terms the necessary limitation of the data, will be discovered.

When

When a question can be so stated as to produce a pure equation, it is generally to be preferred to an affected. Thus the question in the preceeding section by the most obvious notation would produce an affected equation.

II. Solution of Questions producing Quadratic Equations.

The expression of the conditions of the question by equations, or the stating of it, and the reduction likewise of these equations, till we arrive at a quadratic equation involving only one unknown quantity and its square, are effected by the same rules which were given for the solution of simple equations, in Chap. III.

EXAMPLE II.

One lays out a certain sum of money in goods, which he sold again for 24 l. and gained as much per cent. as the goods cost him: I demand what they cost him?

If the money laid out be y

The gain will be $24 - y$
 But this gain is $\left. \begin{array}{l} 2 \\ 3 \end{array} \right\} \frac{2400 - 100y}{y}$ per cent.
 ($y : 24 - y :: 100 :$)

Therefore by question $y = \frac{2400 - 100y}{y}$

And by mult. and tr. $5 y^2 + 100y = 2400$

Completing the square $6 y^2 + 100y + 50^2 = 2400 + 2500 = 4900$

Extr. the root $7 y + 50 = \pm \sqrt{4900} = 70$

Transp. $3 y = \pm 70 - 50 = 20 \text{ or } -120.$

The answer is 20 l. which succeeds. The other root, -120 , has no place in this example.

Any quadratic equation may be resolved also by the general canons at the beginning of this section. That arising from this question, (N^o 5.) belongs to case I. and $a = 100$, $b^2 = 2400$, therefore,

$$y = \left(-\frac{a}{4} \pm \sqrt{\frac{a^2}{4} + b^2} \right) - \frac{100}{2} \pm \sqrt{\frac{100^2}{4} + 2400} = 20$$

or -120 as before.

EXAMPLE III.

What two numbers are those, whose difference is 15, and half of whose product is equal to the cube of the lesser?

Let

Let the lesser number be	1	x
The greater is	2	$x + 15$
By question	3	$x^2 + 15x = x^3$
		$\frac{2}{2}$
Divide by x and mult. by 2	4	$x + 15 = 2x^2$
4th prepared	5	$x^2 - \frac{x}{2} = \frac{15}{2}$
Complete square	6	$x^2 - \frac{x}{2} + \frac{1}{16} = \frac{15}{2} + \frac{1}{16} = \frac{121}{16}$
Ext. $\sqrt{}$	7	$x - \frac{1}{4} = \pm \frac{11}{4}$
transp.	8	$x = 3 \text{ or } -\frac{5}{2}$

The numbers therefore are 3 and 18, which answer the conditions. This is an example of case 2d, and the negative root is useless.

EXAMPLE IV.

To find two Numbers whose sum is 100, and whose product is 2059.

Let the given sum $100 = a$, the product $2059 = b$, and let one of the numbers sought be x , the other will be $a - x$. Their product is $ax - x^2$.

Therefore by question

Complete the square

Ext. $\sqrt{}$

Transp.

And the other number

1	$ax - x^2 = b \text{ or } x^2 - ax = -b$
2	$x^2 - ax + \frac{a^2}{4} = \frac{a^2}{4} - b$
3	$x - \frac{a}{2} = \pm \sqrt{\frac{a^2}{4} - b}$
4	$x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}$
5	$a - x = \frac{a}{2} \mp \sqrt{\frac{a^2}{4} - b}$

By inserting numbers $x = 71$ or 29 and $a - x = 29$ or 71 so that the two numbers sought are 71 and 29 .

Here it is to be observed that b must not be greater than $\frac{a^2}{4}$ else the roots of the equation would be impossible, that is, the given product must not be greater than the square of half the given sum of the numbers sought. This limitation can easily be shewn from other principles, for the greatest possible product of two

two parts, into which any number may be divided, is when each of them is a half of it. If b be equal to $\frac{a^2}{4}$ there is only one solution and $x = \frac{a}{2}$ also $a - x = \frac{a}{2}$.

E X A M P L E V.

There are three numbers in continual geometrical proportion: the sum of the first and second is 10, and the difference of the second and third is 24. What are the numbers?

Let the first be

The second will be

And the third

Since $z : 10 - z : 34 - z$

Transp.

Divid.

Compl. the square

Extract the $\sqrt{}$

Transp.

$$\begin{array}{l|l}
 1 & z \\
 2 & 10 - z \\
 3 & 34 - z \\
 4 & z^2 - 20z + 100 = 34z - z^2 \\
 5 & 2z^2 - 54z = -100 \\
 6 & z^2 - 27z = -50 \\
 7 & z^2 - 27z + \frac{27^2}{4} = \frac{729}{4} - 50 = \frac{529}{4} \\
 8 & z - \frac{27}{2} = \pm \sqrt{\frac{529}{4}} = \pm \frac{23}{2} \\
 9 & z = \frac{27}{2} \pm \frac{23}{2} = 25 \text{ or } 2.
 \end{array}$$

Tho' the preceeding equation is of the third form, yet the data in this question cannot be so related as to make the roots impossible. For if the given numbers 10 and 24 be expressed by a , and b , the equation in step 6th will be $z^2 - \frac{3a+b}{2} \times z = \frac{a^2}{2}$, and whatever be the values of a and b , $\frac{3a+b}{2}$ viz. the square of half the coefficient of z will always be greater than $\frac{a^2}{2}$; and even if b was 0, yet still $\frac{9a^2}{16}$ is greater than $\frac{a^2}{2}$.

But tho' there are two positive roots in this equation, yet one of them only can here be of use, the other being excluded by a condition in the question. For as the sum of the first and second is 10, 25 cannot be one of them: 2 therefore is the first, and the proportionals will be 2, 8, 32.

This restriction will also appear from the explanation given of the third form to which this equation belongs. For z may be less than $\frac{27}{2}$ but from the first condition of the question it cannot

be

be greater ; hence the quantity $z^2 - 27z + \frac{27}{2}$ can have only one square root, viz. $\frac{27}{2} - z$ and this being put equal to $\sqrt{\frac{529}{4}}$

we have by transposition $z = \frac{27}{2} - \frac{23}{2} = 2$ which gives the only just solution of the question.

From the other indeed a solution of the question may be represented by means of a negative quantity. If the first then be 25, the three proportionals will be 25, -15, 9. These also it is obvious, must answer the conditions, according to the rules given for negative quantities.

Besides it is to be observed that if the following question be proposed "to find three numbers in geometrical proportion so that the difference of the 1st and 2d may be 10, and the sum of the 2d and 3d may be 24 : " The equation in step 6th, will be produced : for if the 1st be z , the 2d is $z - 10$, and the 3d $34 - z$ and therefore $34z - z^2 = z^2 - 20z + 100$ the very same equation as in step 4th. In this question it is plain that the root 25 only can be useful, and the three proportionals are 25, 15, 9.

Tho' the preceeding questions have been so contrived that the answers may be integers, yet in practice it will most commonly happen that they must be surds. When in any question the root of a number which is not a perfect square is to be extracted, it may be continued in decimals, by the common arithmetical rule, to any degree of accuracy which the nature of the subject may require.

SCHOLIUM.

An equation in the terms of which, two powers only of the unknown quantity are found, and such that the index of the one is double that of the other, may by the preceeding rules be reduced to a pure equation, and may therefore be resolved by Sect. I. of this chapter.

Such an equation may generally be represented thus,

$$x^{2m} \pm ax^m = \pm b^n$$

Let $x^m = z$, then $z^2 \pm az = \pm b^n$

$$\text{And } x^m (=z) = \pm \frac{a}{2} \pm \sqrt{\frac{a^2}{4} \pm b^n}$$

$$\text{Therefore } x = \sqrt[m]{\pm \frac{a}{2} \pm \sqrt{\frac{a^2}{4} \pm b^n}} \quad \text{EX.}$$

EXAMPLE XV.

To find two numbers, of which the product is 100, and the difference of their square roots 3.

Let the less be x the greater is

By question

$$\begin{array}{l|l}
 1 & \frac{100}{x} \\
 2 & \frac{10}{\sqrt{x}} - \sqrt{x} = 3 \\
 3 & 10 - x = 3\sqrt{x} = 3x^{\frac{1}{2}} \\
 4 & x + 3x^{\frac{1}{2}} = 10 \\
 5 & x + 3x^{\frac{1}{2}} + \frac{9}{4} = 10 + \frac{9}{4} = \frac{49}{4} \\
 6 & x^{\frac{1}{2}} + \frac{3}{2} = \pm \frac{7}{2} \text{ and } x^{\frac{1}{2}} = 2 \text{ or } -5 \\
 7 & x = 4 \text{ or } x = 25
 \end{array}$$

If $x = 4$ the other number is 25, and this is the proper solution for x was supposed to be the least. In this case indeed the negative root of the equation gives a positive answer to the question, and if $x = 25$, the other number is 4.

The same would have been got by substituting in the general theorem, $m = \frac{1}{2}$, $a = 3$, and $b^n = 10$.

C H A P. VI.

ALGEBRA may be employed for the demonstration of Theorems with regard to all those quantities concerning which it may be used as an analysis, and from the general method of notation and reasoning, it possesses the same advantages in the one as in the other. The three first sections of the following chapter contain some of the most simple properties of *series* which are of frequent use; and the last, miscellaneous examples of the properties of algebraical quantities and numbers.

I. Of Arithmetical Series.

Def. When a number of quantities increase or decrease by the same common difference, they form an *Arithmetical Series*.

G

Thus

Thus, $a, a+b, a+2b, a+3b, \&c. x, x-b, x-2b, \&c.$
 Also 1, 2, 3, 4, 5, 6, &c. And 8, 6, 4, 2, &c.

Prop. In an arithmetical series the sum of the first and last terms is equal to the sum of any two intermediate terms equally distant from the extremes.

Let the first term be a , the last x , and b the common difference; then $a+b$ will be the second, and $x-b$ the last but one, &c.

Thus, $a, a+b, a+2b, a+3b, a+4b, \&c.$
 $x, x-b, x-2b, x-3b, x-4b, \&c.$

It is plain, that the terms in the same perpendicular rank are equally distant from the extremes, and that the sum of any two in it is $a+x$, the sum of the first and last.

Cor. 1. Hence the sum of all the terms of an arithmetical series is equal to the sum of the first and last, taken half as often as there are terms.

Therefore if n be the number of terms, and s the sum of the series; $s = a + x \times \frac{n}{2}$. If $a = 0$, then $s = \frac{nx}{2}$.

Cor. 2. Since x must be $a + \overline{n-1} \times b$, b being the common difference, hence also, $s = 2a + \overline{n-1} \times b \times \frac{n}{2}$; and by multiplication, $s = \frac{2an + n^2b - nb}{2}$. Therefore from the first term, the common difference, and number of terms being given, the sum may be found.

Ex. Required the sum of 50 terms of the series 2, 4, 6, 8, &c.

$$s = \frac{2 \times 2 \times 50 + 50^2 \times 2 - 50 \times 2}{2} = \frac{5100}{2} = 2550.$$

Cor. 3. Of the first term, common difference, sum and number of terms, any three being given, the fourth may be found by resolving the preceeding equation; a , b , s , and n being successively considered as the unknown quantity. In the three first cases the equation is simple, and in the last it is quadratic

II. Of Geometrical Series.

Def. When a number of quantities increase by the same multiplier, or decrease by the same divisor, they form a *Geometrical Series*.

Series. This common multiplier or divisor is called the *common ratio*.

Thus $a, ar, ar^2, \&c.$ $a, \frac{a}{r}, \frac{a}{r^2}, \frac{a}{r^3}, \&c.$

Prop. I. The product of the extremes in a geometrical series is equal to the product of any two terms, equally distant from the extremes.

Let a be the first term, y the last; r the common ratio: then the series is,

$$a, ar, ar^2, ar^3, ar^4, \&c.$$

$$y, \frac{y}{r}, \frac{y}{r^2}, \frac{y}{r^3}, \frac{y}{r^4}, \&c.$$

It is obvious, that any term in the upper rank is equally distant from the beginning, as that below it from the end; and the product of any two such is equal to ay , the product of the first and last.

Prob. II. The sum of a geometrical series wanting the first term, is equal to the sum of all but the last term, multiplied by the common ratio.

For in the preceeding series.

$$ar + ar^2 + ar^3, \&c. \dots \dots \dots + \frac{y}{r^3} + \frac{y}{r^2} + \frac{y}{r} + y =$$

$$= r \times a + ar + ar^2, \&c. \dots \dots \dots + \frac{y}{r^4} + \frac{y}{r^3} + \frac{y}{r^2} + \frac{y}{r}.$$

Cor. 1. Therefore s being the sum of the series,

$$s - y \times r = s - a. \quad \text{And } s = \frac{yr - a}{r - 1}.$$

Hence s can be found from a, y , and r , and any three of the four being given, the fourth may be found.

Cor. 2. Since the exponent of r in any term is equal to the number of terms preceeding it, hence in the last term its exponent will be $n - 1$; the last term therefore $y = ar^{n-1}$, and $s = \frac{ar^n - a}{r - 1} = a \times \frac{r^n - 1}{r - 1}$. Hence of these four, s, a, r, n , any

three being given the fourth may be found by the solution of equations. If n is not a small number the cases of this problem will be most conveniently resolved by logarithms; and of such solutions there are examples in the appendix.

Cor. 3. If the series decreases, and the number of terms is infinite,

infinite, then according to this notation a the last term will be 0, and $s = \frac{yr}{r-1}$; a finite sum.

Ex Required the sum of the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \&c.$ to infinity. Here $y = 1$, and $r = 2$. Therefore $s = \frac{1 \times 2}{2 - 1} = 2$.

What are called in arithmetic, repeating and circulating decimals, are truly geometrical decreasing serieses, and therefore may be summed by this rule.

Thus .333, &c. $= \frac{3}{10} + \frac{3}{100} + \&c.$ is a geometrical series in which

$$y = \frac{3}{10} \text{ and } r = 10 \text{ therefore } s = \frac{yr}{r-1} = \frac{3 \times 10}{10 \times 10 - 1} = \frac{1}{3}.$$

Thus also .2424, &c. $= \frac{8}{33}$ for here $y = \frac{24}{100}$ and $r = 100$ therefore

$$s = \frac{24 \times 100}{100 \times 100 - 1} = \frac{24}{99} = \frac{8}{33}.$$

III. Of Infinite Series.

It was observed (Chap. I. and IV.) that in many cases, if the division and evolution of compound quantities be actually performed, the quotients and roots can only be expressed by a series of terms, which may be continued *ad infinitum*. By comparing a few of the first terms, the law of the progression of such a series will frequently be discovered, by which it may be continued without any farther operation. When this cannot be done, the work is much facilitated by several methods; the chief of which is that by the *binomial theorem*.

Prob. To raise any binomial $(a + b)$ to any power (m) .

1. From inspecting a table of the powers of a binomial obtained by multiplication, the terms without their coefficients are $a^m, a^{m-1}b, a^{m-2}b^2, a^{m-3}b^3, \&c.$

2. The coefficients of these terms will be found by the following rule.

Divide the exponent of a in any term by the exponent of b increased by 1, and the quotient multiplied by the coefficient of that term will give the coefficient of the next following term.

This rule is found, upon trial in the table of powers, to hold universally. The coefficient of the first terms is always 1. and by

by applying the general rule now proposed, the coefficients will be as follows $1, m, m \times \frac{m-1}{2}, m \times \frac{m-1}{2} \times \frac{m-2}{3}, \&c.$ They

may be more conveniently expressed thus, $1, Am, B \times \frac{m-1}{2},$

$C \times \frac{m-2}{3}, D \times \frac{m-3}{4}, \&c.$ the capitals denoting the preceding

coefficient. Hence $(a+b)^m = a^m + Ama^{m-1}b + B \times$

$\frac{m-1}{2} \times a^{m-2}b^2 + C \times \frac{m-2}{3} a^{m-3}b^3 \&c.$ This is the

celebrated *binomial theorem*: It is deduced here by induction only, but it may be rigidly demonstrated, tho' upon principles which do not belong to this place.

Cor. As m may denote any number integral or fractional, positive or negative, hence the division, involution and evolution of a binomial may be performed by this theorem.

Ex. 1. Let $m = \frac{1}{2}$, then $(a+b)^{\frac{1}{2}} = a^{\frac{1}{2}} + \frac{1}{2} a^{-\frac{1}{2}} b + \frac{1}{2} \times -\frac{1}{2} \times a^{-\frac{3}{2}} b^2 + \&c.$ This being applied to the extraction of the square root of $a^2 + x^2$ (by inserting a^2 for a and x^2 for b) the same series results as formerly. (Chap. IV.)

Ex. 2. If $\frac{1}{1-r}$ is to be turned into an infinite series, since $\frac{1}{1-r} = 1 \times \frac{1}{1-r} - 1$, let $a = 1, b = -r$, and $m = -1$; and the same series will arise as was obtained by division. (Chap. I.)

In like manner $\frac{r^2}{\sqrt{2rz - z^2}} (= r^2 \times \frac{1}{2rz - z^2} - \frac{1}{2})$ may be expressed by an infinite series by supposing $a = 2rz, b = -z^2$, and $m = -\frac{1}{2}$.

Note, If the binomial is $a + b$, the signs of the terms of any power are all positive: if it is $a - b$, the alternate terms are negative, beginning at the second. This theorem may be applied to quantities which consist of more than two parts, by supposing them distinguished into two, and then substituting for the powers of these compound parts, their values to be obtained also, if required from the theorem.

Thus $(a+b+c)^2 = (a+b+c)^2$.

SCHOLIUM.

An infinite series may itself be multiplied or divided by another

ther ; it may be involved or evolved, and various other operations may be performed upon it which are necessary in the higher parts of algebra. The methods for finding the sum depend upon other principles.

IV. *Properties of Numbers.*

Theor. 1. The sum of the two quantities multiplied by their difference is equal to the difference of their squares.

Let the quantities be represented by a and b then $a + b \times a - b = a^2 - b^2$, as appears by performing the operation.

If a and b be any two quantities of which the sum may be denoted by s , the difference by d , and their product by p , then the following propositions will be true.

$$1. a^2 + b^2 = s^2 - 2p$$

$$2. a^2 - b^2 = sd$$

$$3. a^3 + b^3 = s^3 - 3ps$$

$$4. a^3 - b^3 = s^2d - dp$$

$$5. a^4 + b^4 = s^4 - 4ps^2 + 2p^2$$

$$6. a^4 - b^4 = s^3d - 2sdp, \&c$$

It is unnecessary to express these theorems in words and the demonstrations are very easy by raising $a + b$ to certain powers and making proper substitutions.

Theor. 2. The sum of any number of terms (n) of the odd numbers 1, 3, 5, &c. beginning with 1 is equal to the square of that number (n).

In the rule for summing an arithmetical series, let $a = 1$, $b = 2$ and $n = n$, and the sum of this series will be $s = \frac{2an + n^2b - nb}{2}$

$$= \frac{2n^2}{2} = n^2. \text{ Q. E. D.}$$

Theor. 3. The difference of any two square numbers is equal to the sum of the two roots together with twice the sum of the numbers in the natural scale between the two roots.

Let the one number be p , and the other $p + n$, the intermediate numbers are $p + 1, p + 2, \dots \&c. p + n - 1$. The difference of the squares of the given numbers is $2pn + n^2$; the sum of the two roots is $2p + n$, and twice the sum of the series $p + 1 + p + 2, \dots \&c. p + n - 1$ is by Cor. 1. 1st Sect. $2s = 2p + n \times n - 1$ viz. the sum of the first and last multiplied by the number of terms, and it is plain that $2p + n + 2p + n \times n - 1 = 2pn + n^2$. Therefore, &c.

Lem. 1. Let r be any number and n any integer, $r^n - 1$ is divisible by $r - 1$. The quotient will be $r^{n-1} + r^{n-2} \&c.$ till the

the index of r be 0, and then the last term of it will be 1, for if this series be multiplied by the divisor $r - 1$ it will produce the dividend $r^n - 1$. It will appear also by performing the division and inserting for n any number.

Lem. 2. Let r be any number, and n any integer odd number $r^n + 1$ is divisible by $r + 1$. Also if n is any even number $r^n - 1$ is divisible by $r + 1$. The quotient in both cases is $r^{n-1} - r^{n-2} + r^{n-3} \&c.$ till the exponent of r be 0 and the last term $r^0 = 1$. If this series consist of an odd number of terms and be multiplied by $r + 1$ the divisor, the product is $r^n + 1$ the dividend. If the series consist of an even number of terms, the product is $r^n - 1$. but it is plain that the number of terms will be odd only when n is odd, and even only when n is even. The conclusion will be manifest by performing the division.

Lem. 3. If r is the root of an arithmetical scale, any number in that scale may be represented in the following manner $a, b, c, \&c.$ being the coefficients or digits, $a + br + cr^2 + dr^3 + er^4, \&c.$

Theor. 4. If from any number in the general scale now described the sum of its digits be subtracted the remainder is divisible by $r - 1$.

The number, is, $a + br + cr^2 + dr^3, \&c.$ and the sum of the digits is $a + b + c + d, \&c.$ Subtracting the latter from the former the remainder is $br - b + cr^2 - c + dr^3 - d, \&c. = b \times \frac{r^1 - 1}{r - 1} + c \times \frac{r^2 - 1}{r - 1} + d \times \frac{r^3 - 1}{r - 1} \&c.$ But by Lem. 1 $r^n - 1$ is divisible by $r - 1$ whatever integer number n may be, and therefore any multiple of $r^n - 1$ is also divisible by $r - 1$: hence each of the terms, $b \times \frac{r^1 - 1}{r - 1}, c \times \frac{r^2 - 1}{r - 1} \&c.$ are divisible by $r - 1$ and therefore the whole is divisible by $r - 1$.

Cor. 1. Any number the sum of whose digits is divisible by $r - 1$ is itself divisible by $r - 1$. Let the number be called N and the sum of the digits D ; then by this Prop. $N - D$ is divisible by $r - 1$, and D is supposed to be divisible by $r - 1$: therefore it is plain that N must also be divisible by $r - 1$.

Cor. 2. Any number the sum of whose digits is divisible by an aliquot part of $r - 1$ is also divisible by ~~$r - 1$~~ . For let N and D denote as before and since $N - D$ (Theor. 4.) is divisible by $r - 1$; it is also divisible by an aliquot part of $r - 1$; but D is divisible by an aliquot part of $r - 1$ therefore N is also divisible by that aliquot part. *that part*

*Cor. 3. This theorem with the corollaries relates to any scale whatever. It includes therefore the common property of 9 and

and of 3 its aliquot part, in the decimal scale; for since $r = 10$
 $r - 1 = 9$.

Theor. 5. In any number if from the sum of the coefficients of the odd powers of r the sum of the coefficients of the even powers be subtracted and the remainder added to the number itself, the sum will be divisible by $r + 1$.

In the number $a + br + cr^2 + dr^3 + er^4 + fr^5$, &c. the sum of the coefficients of the odd powers of r is $b + d + f$, &c. the sum of the coefficients of the even powers of r is $a + c + e$, &c. If the latter sum be subtracted from the former and the remainder added to the given number it makes, $br + b + cr^2 - c + dr^3 + d + er^4 - e + fr^5 + f$, &c. $= b \times \overline{r+1} + c \times \overline{r^2-1} + d \times \overline{r+1} + e \times \overline{r^4-1} + f \times \overline{r^5+1}$, &c. But by Lem. 2. $r + 1$, $r^2 - 1$, $r^3 + 1$, &c. are each divisible by $r + 1$ and therefore any multiples of them are also divisible by $r + 1$, hence the whole number is divisible by $r + 1$.

Cor. 1. If the difference of the sum of the even digits and the sum of the odd digits of any number be divisible by $r + 1$ the number itself is divisible $r + 1$.

Let the sum of the even digits (that is the coefficients of the odd powers of r) be D the sum of the odd digits be d and let the number be N . Then by the theorem $N + D - d$ is divisible by $r + 1$ and it is supposed that $D - d$ is divisible by $r + 1$ therefore N is divisible by $r + 1$.

2. In like manner if $D - d$ is divisible by an aliquot part of $r + 1$, N will be divisible by that aliquot part.

Cor. 3. If a number want all the odd powers of r or if it want all the even powers r , and if the sum of its digits be divisible by $r + 1$, that number is divisible by $r + 1$.

Cor. 4. In the common scale $r + 1 = 11$ which therefore will have the properties mentioned in this theorem and the corollaries. Thus in the number 64834, the sum of the even digits is 7, the sum of the odd digits is 18, and the difference is 11, a number divisible by 11, the given number therefore (Cor. 1.) is divisible by 11. Thus also, the sum of the digits of 7040308 is divisible by 11 and therefore the number is divisible by 11. (Cor. 3.)

S C H O L I U M.

These Theorems relate to any scale whatever and therefore the properties of $r - 1$ in Theor. 4. would in a scale of 8 belong to *seven* and those in Theor. 5. to *nine*. If 12 was the root of the scale, the former properties would belong to *eleven* and the latter to *thirteen*.

C H A P. VII.

Of Indeterminate Problems.

IT was formerly observed (Chap. III.) that if there are more unknown quantities in a question than equations by which their relations are expressed, it is indetermined; or it may admit of an infinite number of answers. Other circumstances however, may limit the number in a certain manner, and these are various according to the nature of the problem. The contrivances by which such problems are resolved, are so very different in different cases, that they cannot be comprehended in general rules.

E X A M P L E I.

To divide a given square number into two parts; each of which shall be a square number.

There are two quantities sought in this question, and there is only one equation expressing their relation; another condition therefore must be assumed in such a manner as to obtain a solution in rational numbers.

Let the given square be a^2 , let one of the squares sought be x^2 , the other is $a^2 - x^2$. Let $rx - a$ also be a side of this last square, therefore $r^2x^2 - 2rxa + a^2 = a^2 - x^2$

By transp. $r^2x^2 + x^2 = 2rxa$

Divide by x $r^2x + x = 2ra$

Therefore $x = \frac{2ra}{r^2 + 1}$

$$\text{And } rx - a \left(= \frac{2r^2a}{r^2 + 1} - a \right) = \frac{r^2a - a}{r^2 + 1}$$

Let r therefore be assumed at pleasure, and $\frac{2ra}{r^2 + 1}$, $\frac{r^2a - a}{r^2 + 1}$,

which must always be rational, will be the sides of the two squares required.

Thus, if $a^2 = 100$. Then, if $r = 3$, the sides of the two squares are 6 and 8, for $36 + 64 = 100$.

Also let $a^2 = 64$. Then if $r = 2$, the sides of the squares are

$$\frac{32}{5} \text{ and } \frac{24}{5}; \text{ and } \frac{1024}{25} + \frac{576}{25} = \frac{1600}{25} = 64.$$

H EX.

E X A M P L E II.

To find two square numbers whose difference is given.

Let x^2 and y^2 be the square numbers and a their difference.

$$\text{Put } \frac{z+v}{2} = x, \text{ and } \frac{z-v}{2} = y$$

$$\frac{z^2 + 2zv + v^2}{4} = x^2$$

$$\frac{z^2 - 2zv + v^2}{4} = y^2$$

$$zv = (x^2 - y^2) = a.$$

If x and y are required only to be rational, then take v at pleasure, and $z = \frac{a}{v}$ whence x and y are known.

But if x and y are required to be whole numbers, Take, if possible, any two factors that produce a , and are both even or both odd numbers. And a therefore must be either an odd number greater than 1, or a number divisible by 4.

For the product of two odd numbers is odd, and that of two even numbers is divisible by 4. Also, if z and v are both odd or both even, $\frac{z+v}{2}$ and $\frac{z-v}{2}$ must be integers.

Ex. 1. If $a = 27$, take $v = 1$, then $z = 27$; and the squares are 196 and 169.

2. If $a = 12$ take $v = 2$, and $z = 6$; and the squares are 16 and 4,

E X A M P L E III.

To find a sum of money in pounds and shillings whose half is just its reverse.

Note. The reverse of a sum of money, as 3l. 12s. is 12l. 8s.

Let x be the pounds and y the shillings.

The sum required is $20x + y$

Its reverse is $20y + x$

Therefore $\frac{20x + y}{2} = 20y + x$

$$20x + y = 40y + 2x$$

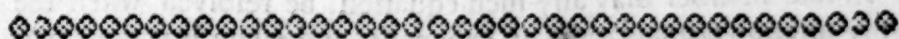
$$18x = 39y$$

$$x : y :: (39 : 18 ::) 13 : 6$$

In this equation there are two unknown quantities; and in general, any two numbers of which the proportion is that of 13 to 6, will agree to it.

But from the nature of this question, 13 and 6 are the only two that can give the proper answer, viz. 13l. 6s. for its reverse 6l. 13s. is just its half.

The ratio of x and y is expressed in the lowest integral terms by 13 and 6; any other expression of it as the next greater 26 and 12, will not satisfy the problem, as 12 l. 26 s. is not a proper notation of money in pounds and shillings.



APPENDIX to Part I.

ALGEBRA may be employed in expressing the relations of magnitude in general, and in reasoning with regard to them. It may be used in deducing not only the relations of number, but also those of extension, and hence, those of every species of quantity expressible by numbers or extended magnitudes. In this appendix are mentioned some examples of its application to other parts of mathematics, to physics and to the practical calculations of business. The principles and suppositions peculiar to these subjects which are necessary in directing, both the algebraical operations and the conclusions to be drawn from them, are here assumed as just and proper.

I.

Algebra has been successfully applied to almost every branch of mathematics, and the principles of these branches are often advantageously introduced into algebraical calculations. The application of it to geometry has been the source of great improvements in both these sciences, a more particular account of which is given in the third part of these elements.

In this place shall be given an example of the use of logarithms in resolving certain algebraical questions.

Note. When Logarithms are used let ($l.$) denote the logarithm of any quantity before which it is placed.

Ex. 1. To find the number of terms of a geometrical series of which the sum is 511, the first term 1, and the common ratio 2.

From Sect. 2. Chap. 6. it appears that $s = \frac{ar^n - a}{r - 1}$, and in

this

this problem, s , r , and a are given and n is to be found. By reducing the equation $r^n = \frac{s \times r - 1 + a}{a}$ and from the known

property of logarithms $n \times l.r = l. \frac{s \times r - 1 + a}{a}$, and $n = \frac{l. \frac{s \times r - 1 + a}{a}}{l. r}$. But here $s = 511$, $a = 1$, $r = 2$, and

$$n = \frac{l. 512}{l. 2} = \frac{2.7092700}{0.3010300} = 9.$$

In like manner may any such equation be resolved when the only unknown quantity is an exponent, and when it is the exponent only of one quantity.

Ex. 2. An equation of the following quadratic form $a^{2x} \pm 2ba^x = \pm c$ may be resolved by logarithms. 1st. by scholium of chap. 5. $a^x = \mp b \pm \sqrt{b^2 \pm c}$. And then x is discovered in the same manner as in the preceeding example. Thus if $a = 2$, $b = 10$, and $c = 96$ and the equation $2^{2x} - 20 \times 2^x = -96$. 1st. $2^x = 10 \pm \sqrt{4} = 12$ or 8 . If $2^x = 8$ then $x = \frac{l. 8}{l. 2} = 3$ and $2^6 - 20 \times 2^3 = -96$ is a true equation. If $2^x = 12$

then $x = \frac{l. 12}{l. 2} = \frac{1.0791812}{0.3010300} = 3.5849$, and this number being inserted for x in the given equation by means of logarithms will answer the conditions.

Ex. 3. The sum of 2000l. has been out at interest for a certain time and 500l. has been at interest double of that time, the whole arrear now due reckoning 4 per cent. compound interest is 6000l. What were the times?

By the rules in the appendix for compound interest it is plain that if $R = 1.04$, and the time at which the 2000l. is at interest be x the arrear of it will be $2000 \times R^x$. The arrear of the 500l. is $500 \times R^{2x}$, hence $500 \times R^{2x} + 2000 \times R^x = 6000$. This resolved gives $R^x = 2$ and $x = \frac{l. 2}{l. R} = 17.67$, + nearly, that is 17 years and 8 months nearly and the double is 35 years and 4 months; which answer the conditions.

II. Examples of Physical Problems.

The solution of such problems must be derived from physical prin-

principles, which are ultimately founded on experience, and are here supposed to be just.

If the principles are inaccurate, the conclusions must in like manner be inaccurate, and if the limits of the inaccuracy in the principles can be ascertained, the corresponding limits in the conclusions deduced algebraically from them may be calculated.

E X A M P L E I.

Let a glass tube, 30 inches (a) long, be filled with mercury, excepting 8 inches (b); and let it be inverted, as in the Torricellian experiment, so that the 8 inches of common air may rise to the top: It is required to find at what height the mercury will remain suspended, the mercury in the barometer being at that time 28 inches (d) high.

The solution of this problem depends upon the following principles:

1. The pressure of the atmosphere is measured by the column of mercury in the barometer; and the elastic force of the air in its natural state which resists this pressure, is therefore measured by the same column.

2. In different states, the elastic force of the air is reciprocally as the spaces which it occupies.

3. In this experiment the mercury which remains suspended in the tube, together with the elastic force of the air in the top of it, being a counterbalance to the pressure of the atmosphere, may therefore be expressed by the column of mercury in the barometer.

Let the mercury in the tube be x inches, the air in the top of it occupies now the space $a-x$; it occupied formerly b inches, and its elastic force was d inches of mercury: Now therefore the force must be $(a-x : b :: d :) \frac{bd}{a-x}$ inches.

Therefore $3d. x + \frac{bd}{a-x} = d$. This reduced and putting

$a + d = 2m$ the equation is $x^2 - 2mx = bd - ad$

This resolved gives $x = m \pm \sqrt{m^2 + bd - ad}$.

In numbers - - $x = 44$ or 14 .

One of the roots 44 is plainly excluded in this case and the other 14 is the true answer. If the experiment could be so contrived that the column suspended be equal to the height of the barometer

barometer d together with the measure of the elastic force of b inches of common air in the space $x - a$, that is, if $x = \frac{bd}{x-a} + d$, or $x - \frac{bd}{x-a} = d$, the equation would be the same as before, and the root 44 would be the true answer.

EXAMPLE II.

The distance of the earth and moon (d), and their quantities of matter (t, l) being given, to find the point of equal attraction, between them,

Let the distance of the point from the earth be x : its distance from the moon will be therefore $d - x$. But gravitation is as the matter directly; and as the square of the distance inversely therefore the earth's attraction is $\frac{t}{x^2}$; and the moon's attraction

$\frac{l}{(d-x)^2}$. But these are here equal, therefore

$$\frac{t}{x^2} = \frac{l}{(d-x)^2}, \text{ and } \frac{\sqrt{t}}{x} = \frac{\sqrt{l}}{d-x}$$

This equation reduced gives $x = \frac{d\sqrt{t}}{\sqrt{t} + \sqrt{l}}$

Or mult. numerator and de- }
nominator by $\sqrt{t} - \sqrt{l}$ } $x = \frac{dt - d\sqrt{tl}}{t - l}$.

In round numbers, let $d = 60$ semidiameters of the earth, $t = 40$, $l = 1$ then $x = 52$ semidiameters nearly. There is another point beyond the moon at which the attractions are equal and it would be found by putting the square root of $(d-x)^2$ to be $x - d$ which in this case would be a positive quantity; and then $x = \frac{dt + d\sqrt{tl}}{t - l} = 72$ nearly. If the quantities had been multiplied before extracting the square roots, the affected quadratic would have given the same two roots.

EXAMPLE III.

Let a stone be dropt into an empty pit; and let the time from the dropping of it to the hearing the scund from the bottom be given: To find the depth of the pit.

Let the time be a ; let the fall of a heavy body in the 1st second

second of time (16.122 feet) be b : also, let the motion of sound in a second (1142 feet) be c .

Let the time of the stones fall be	-	-	-	1	x
The time in which the sound of it moves to the top is				2	$a - x$
The descent of a falling body is as the square of the time therefore the depth of the pit is ($1^2 : x^2 :: b^2$)				3	bx^2
The depth from the motion of sound is also				4	$ca - cx$
Therefore 3 and 4				5	$bx^2 = ca - cx$

This equation being resolved, gives the value of x and from it may be got bx^2 or $ca - cx$, the depth of the pit.

If the time is 10'' then $x = 8\ 89''$ and the depth is 1273 feet.

III. Of Interest and Annuities.

The application of algebra to the calculation of interests and annuities will furnish proper examples of its use in business. Algebra cannot determine the propriety or justice of the common suppositions on which these calculations are founded, but only the necessary conclusions resulting from them.

Notation.

In the following theorems let p denote any principal sum of which 1 l. is the unit, t the time during which it bears interest, of which 1 year shall be the unit, r the rate of interest of 1 l. for 1 year, and let s be the amount of the principal sum p with its interest for the time t at the rate r .

1. Of Simple Interest.

$s = p + ptr$, and of these four, s , p , t , r , any three being given the fourth may be found by resolving a simple equation.

The foundation of the canon is very obvious, for the interest of 1 l. in one year is r , for t years it is tr , and for p pounds it is ptr , the whole amount of principal and interest must therefore be $p + ptr$.

2. Of Compound Interest.

When the simple interest at the end of every year is supposed to be joined to the principal sum, and both to bear interest for the following year, money is said to bear compound interest. The same notation being used, let $1 + r = R$. Then $s = pR^t$.

For the simple interest of 1 l. in a year is r and the new principal sum therefore which bears interest during the second year is

$(1 + r =) R$; the interest of R for a year is rR , and the amount of principal and interest at the end of the 2d year, is $R + rR = R \times 1 + r = R^2$. In like manner at the end of the 3d year it is R^3 , and at the end of t years it is R^t , and for the sum p it is $pR^t = s$.

Cor. 1. Of these four p, R, t, s any three being given the 4th may be found. When t is not very small the solution will be obtained most conveniently by logarithms. When R is known r may be found, and conversely.

Ex. If 500*l.* has been at interest for 21 years, the whole arrear due reckoning $4\frac{1}{2}$ per cent. Compound interest is 1260.12*l.* or 1260*l.* 2*s.* 5*d.* In this case $p = 500, R = 1.045$ and $t = 21$. and $s = 1260.12$, and any one of these may be derived by the theorem from the others being known. Thus to find s ; $\log R^t = t \times \log R = 21 \times 0.0191163 = 0.4014423$, therefore $R^t = 2.520242$ and $s = (pR^t =) 500 \times 2.520242 = 1260.121$.

Cor. 2. The present worth of a sum (s) in reversion that is payable after a certain time t is found thus: let it be x , then this money improved by compound interest during t produces xR^t which must be equal to s , and if $xR^t = s, x = \frac{s}{R^t}$.

Cor. 3. The time in which a sum is doubled at compound interest will be found thus. $pR^t = 2p$ and $R^t = 2$ and $t = \frac{\log 2}{\log R}$; thus if the rate is 5 per cent. $r = .05$ and $\frac{\log 2}{\log 1.05} = \frac{0.3010300}{0.0211893} = 14.2064$. that is 14 years and 75 days nearly.

3. Of Annuities.

An annuity is a payment made annually for a certain term of years, and the chief problem with regard to it is to determine its present worth. The supposition on which the solution proceeds is, that the money received by the seller, being improved by him in a certain manner during the continuance of the annuity, amounts to the same sum as the several payments received by the purchaser, improved in the same manner. The suppositions with regard to the improvement may be various. What is called the method of simple interest, in which simple interests only are reckoned upon the purchase money, and simple interests on each annuity from the time of payment is so manifestly unequitable as to be universally rejected; and the only supposition which is now admitted is the highest improvement possible on both sides, viz. by compound interest. As the taking compound interest is prohibited by law,
by

the realizing of this supposed improvement, requires punctual payment of interest, and therefore, the interest in such calculations is generally made low. Even with this advantage, it can hardly be rendered effectual in its full extent; it is however universally acquiesced in, as the most proper foundation of general rules; and when peculiar circumstances require any different hypothesis, a suitable calculation may be made.

Let then the annuity be called a , and let p be the present worth of it or purchase money, t the time of its continuance, and let the other letters denote as formerly.

The seller by improving the price received p , at compound interest, at the time the annuity ceases has pR^t .

The purchaser receives the first annuity a at the end of the first year, which is improved by him for $t - 1$ years; it becomes therefore (Th. 2.) aR^{t-1} .

He receives the 2d annuity at the end of the 2d year, and when improved $t - 2$ years, it becomes aR^{t-2} .

The third annuity becomes aR^{t-3} , &c.

The last annuity is simply a , therefore the whole amount of the improved annuities is the geometrical series $a + aR + aR^2$, &c... aR^{t-1} . The sum of this series by Chap. 6. Sect. 2.

$$\text{is } a \times \frac{R^t - 1}{R - 1} = a \times \frac{R^t - 1}{r}.$$

But from the nature of the problem $pR^t = a \times \frac{R^t - 1}{r}$. and

$$\text{hence } p = a \times \frac{R^t - 1}{rR^t} = a \times \frac{1 - \frac{1}{R^t}}{r}.$$

Cor. 1. Of these four p , a , R , t , any three being given, the fourth may be found, by the solution of equations; t is found easily by logarithms, R or r can be found only by resolving an affected equation of the t order.

Cor. 2. If an annuity has been unpaid for the term t , the amount reckoning compound interest, will be $a \times \frac{R^t - 1}{r}$.

Cor. 3. The present worth of an annuity in reversion, that is to commence after a certain time (n) and then to continue t years, is found by subtracting the present worth for n years from the present worth for $n + t$ years, and then;

$$p = a \times \frac{R^{n+t} - 1}{rR^{n+t}} = a \times \frac{1 - \frac{1}{R^{n+t}}}{rR^n} \quad \text{I} \quad \text{Also}$$

Also of R , t , n , a , p , any four being given, the fifth may be found.

Cor. 4. If the annuity is to continue for ever then $R^t - 1$ and R^t may be considered as the same; and $p = a \times \frac{R^t - 1}{rR^t} = \frac{a}{r}$.

Cor. 5. A perpetuity in reversion (by Cor. 3.) since $R^t - 1 = R^t$ is $p = \frac{a}{rR^n}$.

Prob. When 12 years of a lease of 21 were expired, a renewal for the same term was granted for 1000l. 8 years are now expired and for what sum must a corresponding renewal be made, reckoning 5 per cent. compound interest?

From the first transaction the yearly profit rent must be deducted; and from this the proper fine in the second may be computed.

In the first bargain an annuity in reversion for 12 years to commence 9 years hence, was sold for 1000l. the annuity will therefore be found by Cor. 3. in which all the quantities are given but $a = p \times \frac{rR^n}{1 - \frac{1}{R^t}}$. And by inserting numbers, viz. $p =$

1000, $t = 12$, $n = 9$, $r = .05$, and $R = 1.05$; and working by logarithms $a = 175.029 = 175\text{l.} - 6\text{d.}$

Next having found a , the second renewal is made by finding the present worth of the annuity a in reversion, to commence 13 years hence and to last 8 years. In the canon (Cor. 3.) insert for a , 175.029 and let $t = 8$, $n = 13$, and $r = .05$ as before. $p = 599\ 93 = 599\text{l.} 18\text{s.} 6\text{d.} \frac{1}{2}$. The fine required.

As these computations often become troublesome, and are of frequent use, all the common cases are calculated in tables, from which the value of any annuity, for any time, at any interest, may easily be found.

The calculations of life annuities, depend partly on the principles now explained, and partly on physical principles, from the value of human life as deduced from bills of mortality.

E L E M E N T S O F A L G E B R A.

P A R T II.

Of the General Properties and Resolution
of EQUATIONS of all Degrees.

C H A P. I.

*Of the Genesis of Equations, and of the
Signs and Coefficients of their Terms.*

IN order to resolve the higher orders of equations, and to investigate their general affections, it is proper first to consider their origin from the combination of inferior equations.

As it would be impossible to exhibit particular rules for the solution of every order of equations, their number being indefinite; there is a necessity of deducing rules from their general properties which may be equally applicable to all.

In the application of algebra to certain subjects, and especially to geometry, there may be an opposition in the quantities, analogous to that of addition and subtraction, which may therefore be expressed by the signs $+$ and $-$. Hence these signs may be understood by abstraction, to denote contrariety in general; and therefore in this method of treating of equations, negative roots are admitted as well as positive. In many cases, the negative roots will have a proper and determinate meaning, and when the equation relates to magnitude only, where contrariety cannot be supposed to exist, these roots are neglected, as in the case of quadratic equations formerly explained. There is besides this advantage in admitting negative roots, that both the properties of equations from which their resolution is obtained, and also those
which

which are useful in the many extensive applications of algebra, become more simple and general, and are more easily deduced.

In this general method all the terms of any equation are brought to one side, and the equation is expressed by making them equal to 0. Therefore if a root of the equation be inserted instead of (x) the unknown quantity, the positive terms will be equal to the negative, and the whole must be equal to 0.

Def. When any equation is put into this form, the term in which (x) the unknown quantity is of the highest power, is called the *First*; that in which the index of x is less by 1, is the *Second*, and so on, till the last into which the unknown quantity does not enter, and is called the *Absolute Term*.

Prop. I. If any number of equations be multiplied together, an equation will be produced, of which the dimension is equal to the sum of the dimensions of the equations multiplied.

If any number of simple equations be multiplied together, as $x - a = 0$, $x - b = 0$, $x - c = 0$, &c. it is obvious, that the product will be an equation of as many dimensions as there are simple equations. In like manner, if higher equations are multiplied together, as a cubic and quadratic, one of the fifth order is produced, and so on.

Conversely. An equation of any dimension is considered as compounded either of simple equations, or of others such that the sum of their dimensions is equal to the dimension of the given one; and by the resolution of equations these inferior equations are discovered.

Cor. 1. Any equation admits of as many solutions, or has as many roots, as there are simple equations which compose it, that is, as there are units in the dimension of it.

Cor. 2. And conversely, no equation can have more roots than the units in its dimension.

Cor. 3. Imaginary or impossible roots must enter an equation by pairs; for they arise from quadratics, in which both the roots are such.

Cor. 4. The roots are either positive or negative, according as the roots of the simple equations, from which they are produced, are positive or negative.

Prop. II. To explain the general properties of the signs and coefficients of the terms of an equation.

Let $x - a = 0$, $x - b = 0$, $x - c = 0$, $x - d = 0$, &c. be any number of simple equations, of which the roots are any quantities, a , b , c , d , &c. and let them be multiplied as in the following table.

$x - a$

$$x - a = 0$$

$$\times x - b = 0$$

$$\begin{array}{r} = x^2 - ax \\ - bx + ab \end{array} \} = 0, \text{ a Quadratic.}$$

$$\times x - c = 0$$

$$\begin{array}{r} = x^3 - a \\ - b \\ - c \end{array} \} \times x^2 \begin{array}{r} + ab \\ + ac \\ + bc \end{array} \} \times x - abc = 0, \text{ a Cubic.}$$

$$\times x - d = 0$$

$$\begin{array}{r} = x^4 - a \\ - b \\ - c \\ - d \end{array} \} \times x^3 \begin{array}{r} + ab \\ + ac \\ + ad \\ + bc \\ + bd \\ + cd \\ \&c. \end{array} \} \times x^2 \begin{array}{r} - abc \\ - abd \\ - acd \\ - bcd \end{array} \} \times x + abcd = 0, \text{ a Biquadra-} \\ \text{tic,}$$

From this table it is plain,

1. That in a complete equation the number of terms is always greater by unit than the dimension of the equation.

2. The coefficient of the first term is 1.

The coefficient of the second term is the sum of all the roots ($a, b, c, d, \&c.$) with their signs changed.

The coefficient of the third term is the sum of all the products that can be made by multiplying any two of the roots together.

The coefficient of the fourth term is the sum of all the products which can be made by multiplying together any three of the roots with their signs changed; and so of others.

The last term is the product of all the roots, with their signs changed.

3. From induction it appears, that in any equation (the terms being regularly arranged as in the preceeding example) there are as many positive roots as there are changes in the signs of the terms from $+$ to $-$, and from $-$ to $+$; and the remaining roots are negative.

Note. The impossible roots in this rule are supposed to be either positive or negative.

In this example of a numeral equation $x^4 - 10x^3 + 35x^2 - 50x + 24$

$+24=0$, the roots are, $+1, +2, +3, +4$, and the preceeding observations with regard to the signs and coefficients take place.

Cor. If a term of an equation is wanting, the positive and negative parts of its coefficient must then be equal. If there is no absolute term, then some of the roots must be $= 0$, and the equation may be depressed by dividing all the terms by the lowest power of the unknown quantity in any of them.

C H A P. II.

Of the Transformation of Equations.

THere are certain transformations of equations necessary towards their solution: the most useful are contained in the following propositions.

Prop. I. The affirmative roots of an equation become negative, and the negative become affirmative, by changing the signs of the alternate terms, beginning with the second.

Thus the roots of the equation $x^4 - x^3 - 19x^2 + 49x - 30 = 0$ are $+1, +2, +3, -5$, whereas the roots of the equation, $x^4 + x^3 - 19x^2 - 49x - 30 = 0$, are $-1, -2, -3, +5$.

The reason of this is easily derived from the composition of the coefficients of these terms, which consist of combinations of odd numbers of the roots, as explained in the preceeding chapter.

Prop. II. An equation may be transformed into another that shall have its roots greater or less than the roots of the given equation by some given difference.

Let x be the unknown quantity of the equation, and e the given difference: let $y = x \pm e$, then $x = y \mp e$; and if for x and its powers in the given equation, $y \mp e$ and its powers be inserted, a new equation will arise, in which the unknown quantity is y , and its value will be $x \pm e$; that is, its roots will differ from the roots of the given equation by e .

Let the equation proposed be $x^3 - px^2 + qx - r = 0$, of which the roots must be diminished by e . By inserting for x and its powers, $y + e$ and its powers, the equation required is,

$$\left. \begin{array}{l} y^3 + 3ey^2 + 3e^2y + e^3 \\ - py^2 - 2pey - pe^2 \\ + qy + qe \\ - r \end{array} \right\} = 0.$$

Cor.

Cor 1. From this transformation the second, or any other intermediate term may be taken away; granting the resolution of equations.

Since the coefficients of all the terms of the transformed equation, except the first, involve the powers of e and known quantities only, by putting the coefficient of any term equal to 0, and resolving that equation, a value of e may be determined; which being substituted, will make that term to vanish.

Thus in this example, to take away the second term, let its coefficient, $3e - p = 0$, and $e = \frac{1}{3}p$, which being substituted for e , the new equation will want the second term. And universally the second term of a cubic equation may be taken away by supposing $x = y \mp \frac{1}{3}p$, $\pm p$ being the coefficient of that term.

Cor. 2. The second term may be taken away by the solution of a simple equation, the third by the solution of a quadratic; and so on.

Cor. 3. If the second term of a quadratic equation be taken away, it will become a pure equation, and thus a solution of quadratics will be obtained which coincides with the solution already given in Part. I.

Cor. 4. The last term of the transformed equation is the same with the given equation, only having e in place of x .

Prop. III. In like manner may an equation be transformed into another, of which the roots shall be equal to the roots of the given equation, multiplied or divided by a given quantity.

Let x be the unknown letter in the given equation, and y that of the equation wanted; also let e be the given quantity.

To multiply the roots let $xe = y$, and $x = \frac{y}{e}$.

To divide the roots let $\frac{x}{e} = y$, and $x = ye$.

Then substitute for x and its powers, $\frac{y}{e}$ or ye , and its powers, and the new equation of which y is the unknown quantity, will have the property required.

Cor. By this proposition, fractions and sometimes furds may be taken out of an equation.

CHAP.

C H A P. III.

Of the Resolution of Equations.

FROM the preceeding principles and operations, rules may be derived for resolving equations of all orders.

I. CARDAN'S Rule for Cubic Equations.

The second term of a cubic equation being taken away (by Cor. 1. Prop. 2. Chap. II.) it may be generally represented by $x^3 + 3qx + 2r = 0$. The sign + in all the terms, denotes the addition of them with their proper signs. Let $x = m + n$, and also $mn = -q$; by the substitution of these values, an equation of the 6th order, but of the quadratic form is deduced, which gives the values of m and n , and hence;

$$(m + n =) x = \sqrt[3]{-r + \sqrt{r^2 + q^3}} + \sqrt[3]{-r - \sqrt{r^2 + q^3}}.$$

Cor. 1. In the given equation if $3q$ is negative, and if r^2 is less than q^3 , this expression of the root involves impossible roots; while at the same time, all the roots of that equation are possible. The reason is, that in this method of solution it is necessary to suppose that x the root, may be divided into two parts, of which the product is q . But it is easy to shew that in this which is called the *irreducible case*, it cannot be done.

If the cube root of the compound surd can be extracted, the impossible parts balance each other, and the true root is obtained.

The geometrical problem of the trisection of an arch is resolved algebraically, by a cubic equation of this form; and hence the foundation of the rule for resolving an equation belonging to this case by a table of sines.

Cor. 2. Biquadratic equations may be reduced to cubics, and may therefore be resolved by this rule.

Some other classes of equations too, may be resolved by particular rules; but these, and every other order of equations, are commonly resolved by the general rules, which may be equally applied to all.

It is to be observed in general that when one root of an equation is discovered, one of the simple equations is found, from which the given one is compounded. The given equation therefore being divided by this simple one, will give an equation of a dimension lower than it by 1. Thus, any equation may be de-
pressed

pressed as many degrees as there are roots found by any method whatever.

II. Solution of Equations, whose Roots are commensurate.

R U L E I.

All the terms of the equation being brought to one side, find all the divisors of the absolute term, and substitute them successively in the equation for the unknown quantity. That divisor which substituted in this manner gives the result $= 0$, shall be a root of the equation.

$$\text{Ex. 1. } \left. \begin{array}{l} x^3 - 3ax^2 + 2a^2x - 2a^2b \\ - bx^2 + 3abx \end{array} \right\} = 0.$$

The simple literal divisors of $-2a^2b$, are $a, b, 2a, 2b$, any of which may be inserted for x . Supposing $x = +a$, the equation becomes

$$\left. \begin{array}{l} a^3 - 3a^3 + 2a^3 - 2a^2b \\ - ba^2 + 3a^2b \end{array} \right\} \text{ which is obviously } = 0.$$

$$\text{Ex. 2. } x^3 - 2x^2 - 33x + 90 = 0.$$

The divisors of 90 are 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90.

The first of these divisors, which being inserted for x , will make the result $= 0$ is $+3$; $+5$ is another, and it is plain the last root must be negative, and it is -6 .

When 3 is discovered to be a root, the given equation may be divided by $x - 3 = 0$, and the result will be a quadratic, which being resolved will give the other two roots, $+5$ and -6 .

The reason of the rule appear from the property of the absolute term formerly defined, viz. that it is the product of all the roots.

To avoid the inconvenience of trying many divisors, this method is shortened by the following

R U L E II.

Substitute in place of the unknown quantity successively the terms of the progression, 1, 0, -1 , &c. and find all the divisors of the sums that result; then take out all the arithmetical progressions that can be found among these divisors whose common difference is 1, and the values of x will be among those terms of the progressions which are the divisors of the result arising from the substitution of $x = 0$. When the series increases, the roots will be positive; and when it decreases, the roots will be negative.

E X A M P L E

Let it be required to find a root of the equation $x^3 - x^2 - 10x + 6 = 0$.

K

The

The operation is thus :

<i>Supposit.</i>		<i>Result.</i>	<i>Divisors</i>	<i>Arith prog.</i>
$x = 1$	} $x^3 - x^2 - 10x + 6 =$	- 4	1, 2, 4	4
$x = 0$		+ 6	1, 2, 3, 6	3
$x = -1$		+ 14	1, 2, 7, 14	2

In this example there is only one progression, 4, 3, 2, and therefore 3 is a root, and it is - 3, since the series decreases.

It is evident from the rules for transforming equations (Chap. 2.) that by inserting for x , $+ 1 (= + e)$ the result is the absolute term of an equation of which the roots are less than the roots of the given equation by 1 ($= e$). Cor. 4. Prop. 2. When $x = 0$ the result is the absolute term of the given equation. When for x is inserted $- 1 (= - e)$ the result is the absolute term of an equation whose roots exceed the roots of the given equation by 1 ($= e$). Hence if the terms of the series 1, 0, - 1, - 2, &c. be inserted successively for x , the results will be the absolute terms of so many equations, of which the roots form an increasing arithmetical series with the difference 1. But as the commensurate roots of these equations must be among the divisors of their absolute terms, hence they must be among the arithmetical progressions found by this rule. The roots of the given equation therefore are to be sought for among the terms of these progressions which are divisors of the result, upon the supposition of $x = 0$, because that result is its absolute term.

It is plain that the progressions must always be increasing, only it is to be observed that a decreasing series with the sign $+$ becomes increasing with the sign $-$. Thus in the preceding example - 4, - 3, - 2, is an increasing series of which - 3 is to be tried, and it succeeds.

III. Examples of Questions producing the higher Equations.

E X A M P L E I.

It is required to divide 16l. between two persons, so that the cube of the one's share may exceed the cube of the other's by 386.

Let the greater share be x pounds,

And the less will be $16 - x$;

By the question, $x^3 - \overbrace{16 - x}^3 = 386$

And by Inv. $2x^3 - 48x^2 + 768x - 4096 = 386$

Transp. and divide $x^3 - 24x^2 + 384x - 2241 = 0$.

Suppos.

<i>Suppos.</i>	<i>Results.</i>	<i>Divisors.</i>
If $x = 1$; - - -	1880 - - -	1, 2, 4, 5, 8, 10, 20,
$x = 0$; - - -	2241 - - -	1, 3, 9, 27, 83.
$x = -1$; - - -	2650 - - -	1, 2, 5, 10, 25, 53,

Where 8, 9, 10, differ by 1; therefore + 9 is to be tried; and being inserted for x , the equation is $= 0$. The two shares then are 9 and 7 which succeed. Since $x = 9$; $x - 9 = 0$, is one of the simple equations from which this cubic is produced; therefore

$$\frac{x^3 - 24x^2 + 384x - 2241}{x - 9} = x^2 - 15x + 249 = 0. \text{ And the}$$

two roots of this quadratic are impossible.

E X A M P L E II.

What two numbers are those, whose product multiplied by the greater will produce 405, and their difference multiplied by the less 20?

Let the greater Number be x , and the less, y .

$$\text{Then by quest. } \begin{cases} (xy \times x =) x^2y = 405 \\ (x - y \times y =) xy - y^2 = 20 \end{cases}$$

$$\text{Therefore } - - - x = \frac{y^2 + 20}{y}$$

$$\text{And } - - - x^2 = \frac{y^4 + 40y^2 + 400}{y^2}$$

$$\text{Also } - - - x^2 = \frac{405}{y}$$

$$\text{Therefore } \frac{y^4 + 40y^2 + 400}{y^2} = \frac{405}{y}$$

$$\text{Mult. and transp. } y^4 + 40y^2 - 405y + 400 = 0.$$

This biquadratic resolved by divisors, gives $y = 5$: and therefore $x = 9$. Also $\frac{y^4 + 40y^2 - 405y + 400}{y - 5} = y^3 + 5y^2 + 65y - 80 = 0.$

This cubic equation has one positive incommensurate root, viz. 1.114, &c. which may be found by the rule in the next section, and two impossible. The incommensurate root $y = 1.114$, &c. gives $x = 19.067$, &c. and these two answer the conditions very nearly

E X A M P L E III.

The sum of the squares of two numbers 208, and the sum of their cubes 2240 being given, to find them.

Let the greater be $x + y$ and the less $x - y$.

Then $(x + y)^3 + (x - y)^3 = 2x^3 + 2y^3 = 208$

$$\text{Hence } y^3 = 104 - x^3$$

Also $(x + y)^3 + (x - y)^3 = 2x^3 + 6xy^2 = 2240$

Substitute for y^3 its value and $2x^3 + 624x - 6x^3 = 2240$

This reduced gives $x^3 - 156x + 560 = 0$.

The roots of this equation are $+10, +4, -14$. If $x = 10$ then $y = 2$, and the numbers sought are 12 and 8, which give the only just solution. If $x = 4$, then $y^3 = 88$ and $y = \sqrt[3]{88}$. The numbers sought are therefore $4 + \sqrt[3]{88}$ and $4 - \sqrt[3]{88}$. The last is negative but they answer the conditions. Lastly if $x = -14$, then $y^3 = -92$, hence $y = \sqrt[3]{-92}$, is impossible, but still the two numbers $-14 + \sqrt[3]{-92}$, $-14 - \sqrt[3]{-92}$, being inserted would answer the conditions.

IV. *Solution of Equations by Approximation.*

By the former rules, the roots of equations when they are commensurate may be obtained: these however more rarely occur; and when they are incommensurate, we can find only an approximate value of them, but to any degree of exactness required. There are various rules for this purpose; one of the most simple, is that of Sir Isaac Newton, which will be easily understood by the following example.

Lemma. If any two numbers being inserted for the unknown quantity (x) in any equation, give results with opposite signs, an odd number of roots must be between these numbers.

This appears from the property of the absolute term, and from this obvious maxim, that if a number of quantities be multiplied together, and if the signs of an odd number of them be changed, the sign of the product is changed. For when a positive quantity is inserted for x the result is the absolute term of an equation whose roots are less than the roots of the given equation by that quantity. (Prop. 2. Cor 3. Chap. 2.) If the result has the same sign, as the given absolute term, then from the property of the absolute term (prop. 2. Chap. 1.) either none or an even number only of the positive roots, have had their signs changed by the transformation; but if the result has an opposite sign to that of the given absolute term, the signs of an odd number of the positive roots must have been changed. In the first case then the quantity substituted must have been either greater than each of an even number of the positive roots of the given equations, or less than any

any of them ; in the second case it must have been greater than each of an odd numbers of the positive roots. An odd number of the positive roots therefore must lye between them when they give results with opposite signs. The same observation is to be extended to the substitution of negative quantities and the negative roots.

Note, If two adjacent numbers in the scale give opposite results, one root only lyes between them, if there are no decimal fractions in any of the terms.

Let the equation be $x^3 - 2x - 5 = 0$.

1. By trials find the nearest integer to the root ; in this case the root is between 2, and 3, for these numbers being inserted for x , the one gives a positive, and the other a negative result.

2. Suppose $x = 2 + f$ and substitute this value of x in the equation.

$$\begin{array}{r} x^3 = 8 + 12f + 6ff + f^3 \\ - 2x = -4 - 2f \\ - 5 = -5 \\ \hline x^3 - 2x - 5 = -1 + 10f + 6f^2 + f^3 = 0. \end{array}$$

As f is less than unit, its powers f^2 and f^3 may be neglected in this first approximation, and $10f = 1$, or $f = 0.1$ nearly, therefore $x = 2.1$ nearly.

3. As $f = 0.1$, nearly, let $f = .1 + g$, and insert this value of f in the preceeding equation.

$$\begin{array}{r} f^3 = 0.001 + 0.03g + 0.3g^2 + g^3 \\ 6f^2 = 0.06 + 1.2g + 6.g^2 \\ 10f = 1 + 10g \\ - 1 = -1 \\ \hline \end{array}$$

$$f^3 + 6f^2 + 10f - 1 = 0.061 + 11.23g + 6.3g^2 + g^3 = 0$$

and neglecting g^2 and g^3 as very small $0.061 + 11.23g = 0$

$$\text{or } g = \frac{-0.061}{11.23} = -.0054, \text{ hence } f = 0.1 + g = .0946 \text{ nearly}$$

and $x = 2.0946$ nearly.

4. This operation may be continued to any length, as by supposing $g = -.0054 + h$ and so on, and the value of $x = 2.09455147$ nearly.

By the first operation a nearer value of x may be found thus ; since $f = .1$ nearly and $-1 + 10f + 6f^2 + f^3 = 0$,

$$f = \frac{1}{10 + 6f + f^2} \text{ that is } f = \frac{1}{10 + .6 + .01} = .094 \text{ true to the}$$

the last figure, and $x = 2.094$. In the same manner may the root of a pure equation be found, and this gives an easy method of approximating to the roots of numbers which are not perfect powers.

This rule is applicable to numeral equations of every order, and by assuming a general equation, general rules may be deduced for approximating to the roots of any proposed equation. By a similar method we may approximate to the roots of literal equations which will be expressed by infinite series.



E L E M E N T S O F A L G E B R A.

P A R T III.

Of the Application of Algebra to Geometry.

C H A P. I.

General Principles.

GEOMETRY treats both of the magnitude and position of extension, and their connections.

Algebra treats only of magnitude. Therefore, of the relations which subsist in geometrical figures, those of magnitude only can be immediately expressed by algebra.

The

The opposite position of straight lines may indeed be expressed simply by the signs $+$ and $-$. But in order to express the various other positions of geometrical figures by algebra, from the principles of geometry, some relations of magnitude must be found, which depend upon these positions, and which can be exhibited by equations: and conversely, by the same principles, may the positions of figures be inferred from the equations denoting such relations of their parts.

Tho' this application of algebra appears to be indirect, yet such is the simplicity of the operations, and the general nature of its theorems, that investigations, especially in the higher parts of geometry, are generally easier and more expeditious by the algebraical method, tho' less elegant than by what is purely geometrical. The connections also, and analogies of the two sciences established by this application, have given rise to many curious speculations; geometry has been rendered far more extensive and useful, and algebra itself has received considerable improvements.

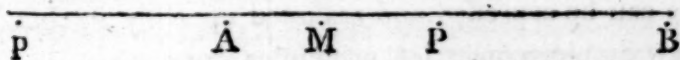
I. Of the Algebraical Expression of Geometrical Magnitudes.

A line, whether known or unknown, is represented by a single letter; a rectangle is expressed by the product of the two letters representing its sides; and a rectangular parallelepiped by the product of three letters, two of which represent the sides of any of its rectangular bases, and the third the altitude.

These are the most simple expressions of geometrical magnitudes, and any other which has a known proportion to these, may in like manner be expressed algebraically. Conversely, the geometrical magnitudes represented by such algebraical quantities may be found, only the dimensions above the third, not having any corresponding geometrical dimensions, must be expressed by proportionals.

The opposite position of straight lines, it has been remarked, may be expressed by the signs $+$ and $-$.

Thus let a point A be given in the line AB, any segment AP



taken to the right hand being considered as positive, a segment AP to the left is properly represented by a negative quantity. If a and b represent two lines, and if upon the line AB from the point A, AP be taken towards the right equal to a , it may be ex-

expressed by $+a$; then PM taken to the left and equal to b , will be properly represented by $-b$, for AM is equal to $a-b$: If $a=b$, then M will fall upon A, and $a-b=0$: by the same notation if b is greater than a , M will fall to the left of A, and in this case if $2a=b$, and if Pp be taken equal to b , then $(a-b=) -a$ will represent Ap which is equal to a , and situated to the left of A.

The positions of geometrical figures are so various, that it is impossible to give general rules for the algebraical expression of them. The following are a few examples.

An angle is expressed by the ratio of its sine to the radius; a right angle in a triangle by putting the squares of the two sides equal to the square of the hypotenuse; the position of points is ascertained by the perpendiculars from them on lines given in position; the position of lines by the angles which they make with given lines, or by the perpendiculars on them from given points: the similarity of triangles by the proportionality of their sides which gives an equation, &c.

These and other geometrical principles must be employed both in the demonstration of theorems, and in the solution of problems. The geometrical proposition must first be expressed in the algebraical manner, and the result after the operation must be expressed geometrically.

II. *The Demonstration of Theorems.*

All propositions in which the proportions of magnitudes only are employed, also all propositions expressing the relations of the segments of a straight line, of their rectangles, squares, and higher powers are demonstrated algebraically with great ease: such demonstrations indeed, may in general be considered as an abridged notation of what are purely geometrical. The II. Book of Euclid.

If certain positions are either supposed or to be inferred in a theorem, we must find according to the preceding observations, the connection between these positions and such relations of magnitude as can be expressed and reasoned upon by algebra.

From few simple geometrical principles alone, a number of conclusions with regard to figures may be deduced by algebra; and to this in a great measure is owing the extensive use of this science in geometry. If other more remote geometrical principles are occasionally introduced, the algebraical calculations may be much abridged. The same is to be observed in the solution of
pro-

problems; but such in general are less obvious, and more properly belong to the strict geometrical method.

III. *The Solution of Problems.*

Upon the same principles, are geometrical problems to be resolved. The problem is supposed to be constructed, and proper algebraical notations of the known and unknown magnitudes are to be sought for, by means of which their connections may be expressed by equations. No general rules can be given on this subject, but the following observations may be of use.

1. The construction of the problem being supposed, it is often farther necessary to produce some of the lines till they meet; to draw new lines joining remarkable points, to draw lines from such points perpendicular or parallel to other lines, and such other operations as seem conducive to the finding of equations; and for this purpose those especially are to be employed, which divide the scheme into triangles that are given, right angled or similar.

2. It is often convenient to denote by letters, not the quantities particularly sought but some others from which they can easily be deduced. The same may be observed of given quantities.

3. The proper notation being made, the necessary equations are to be derived by the use of the most simple geometrical principles, such as the addition and subtraction of lines or of squares, the proportionality of lines, particularly of the sides of similar triangles, &c.

4. There must be as many independent equations, as there are unknown quantities assumed in the investigation, and from these a final equation is to be inferred by the preceeding rules.

By inserting numbers for the known quantities, a numeral expression of the quantities sought will be obtained by resolving the equation. But in order to determine some particulars of the problem, besides finding the unknown quantities of the equation, it may be farther necessary to make a simple construction; or if it is required that every thing be expressed in numbers, to substitute a new calculation in place of that construction.

If the final equation from the problem is simple or quadratic, the roots being obtained by the common rules, may be geometrically exhibited by the finding of proportionals, and the addition or subtraction of squares.

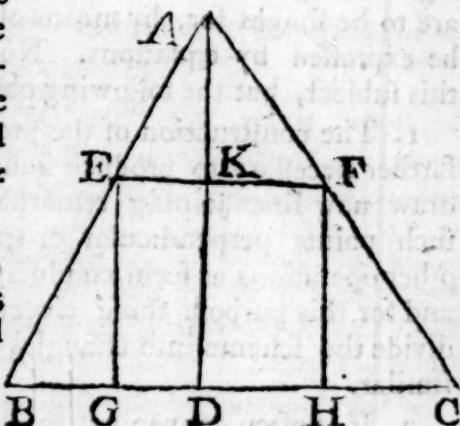
P R O B. I.

In a given Triangle ABC to inscribe a Square.

Suppose it to be done, and let it be EFHG; from A let AD be perpendicular on the base BC, meeting EF in K.

Let $BC = a$, and $AD = p$, both of which are given because the triangle is given: let AK be assumed as the unknown quantity, because from it the square can easily be constructed, and let it be called x : then ($KD = EG =$) $EF = p - x$.

On account of the parallels EF, BC , $AD : BC :: AK : EF$; that is $p : a :: x : p - x$ and $p^2 - px = ax$, which equation being resolved gives $x = \frac{p^2}{p + a}$.

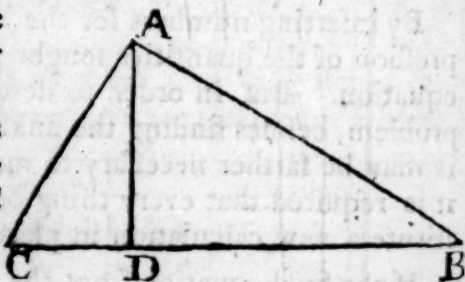


Therefore x or AK is a third proportional to $p + a$, and p , and may be found by 11. VI. El. The point K being found, the construction of the square is sufficiently obvious.

P R O B. II.

In the right-angled Triangle ABC, the Base BC and the Sum of the Perpendicular and Sides $BA + AC + AD$ being given, to find the Triangle.

Such parts of this triangle are to be found as are necessary for describing it: the perpendicular AD will be sufficient for this purpose, and let it be called x : let $AB + AC + AD = a$, $BC = b$, therefore $BA + AC = a - x$: let $BA - AC$ be de-



noted by y , then $BA = \frac{a + y - x}{2}$, and $AC = \frac{a - x - y}{2}$. But

[47. I. El.] $BC^2 = BA^2 + AC^2$ which being expressed algebraically, be-

becomes, $b^2 = \frac{a+y-x}{2}^2 + \frac{a-x-y}{2}^2 = \frac{a^2 - 2ax + x^2 + y^2}{2}$

Likewise from a known property of right-angled triangles, $BC \times AD = BA \times AC$; that is $bx = \left(\frac{a+y-x}{2} \times \frac{a-x-y}{2} \right) = \frac{a^2 - 2ax + x^2 - y^2}{4}$. This last equation being multiplied by 2

and added to the former, gives $b^2 + 2bx = a^2 - 2ax + x^2$, which being resolved according to the rules of Part I. Chap. 5. gives $x = a + b - \sqrt{2ab + 2b^2}$.

To construct this: $a + b$ is the sum of the perimeter and perpendicular, and is given; $\sqrt{2ab + 2b^2} = \sqrt{a + b \times 2b}$ is a mean proportional between $a + b$ and $2b$, and may be found; therefore from the sum of the perimeter and perpendicular, subtract the mean proportional between the said sum, and double the base, and the remainder will be the perpendicular required.

From the base and perpendicular, the right-angled triangle is easily constructed.

In numbers, let $BA + AC + AD = 18.8 = a$; $BC = 10 = b$; then $AD = a + b - \sqrt{2ab + 2b^2} = 18.8 - \sqrt{576} = 4.8 = x$. By either of the first equations $y^2 = 2b^2 + 2ax - a^2 - x^2 = 4$ and $y = 2$; therefore $BA = 8$, and $AC = 6$.

The geometrical expression of the roots of final equations arising from problems, may be found without resolving them, by the intersection of geometrical lines. Thus the roots of a quadratic are found by the intersections of the circle and straight line, those of a cubic and biquadratic, by the intersections of two conic sections, &c.

The solution of problems may be effected also by the intersections of the loci of two intermediate equations without deducing a final equation: but these two last methods can only be understood by the doctrine of the loci of equations.

CH A P. II.

Of the Definition of Lines by Equations.

Lines which can be geometrically treated of, must be produced according to an uniform rule, which determines the position of every point of them.

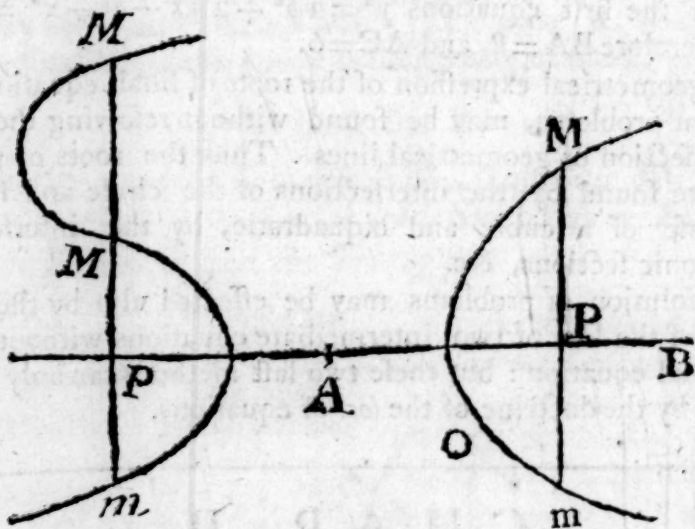
The

This rule constitutes the definition of any line from which all its other properties are to be derived.

A straight line has been considered as so simple, as to be incapable of definition. The curve lines here treated of, are supposed to be in a plane, and are defined either from the section of a solid by a plane, or more universally by some continued motion in a plane, according to particular rules. Any of the properties which are shewn to belong peculiarly to such a line, may be assumed also as the definition of it, from which all the others, and even what upon other occasions may have been considered as the primary definition, may be demonstrated. Hence lines may be defined in various methods, of which the most convenient is to be determined by the purpose in view. The simplicity of a definition, and the ease with which the other properties can be derived from it, generally give a preference.

DEFINITIONS.

I. When curve lines are defined by equations, they are supposed to be produced by the extremity of one straight line, as PM moving in a given angle along another straight line AB given in position, which is called the *Base*.



II. The straight line PM moving along the other, is called an *Ordinate*, and is usually denoted by y .

III. The segment of the base AP between a given point in it A and an ordinate PM is called an *Absciss* with respect to that ordinate, and is denoted by x . The ordinate and absciss together are called *Co-ordinates*. IV.

IV. If the relation of the variable absciss and ordinate AP and PM, be expressed by an equation, which besides x and y contains only known quantities, the curve MO described by the extremity of the ordinate in moving along the base, is called the *Locus* of that equation.

V. If the equation is finite, the curve is called *Algebraical*. It is this class only which is here considered.

VI. The dimensions of such equations are estimated from the highest sum of the exponents of x and y in any term.—According to this definition the terms x^4 , x^3y , x^2y^2 , xy^3 , y^4 are all of the same dimension.

VII. Curve lines are divided into *orders* from the dimensions of their equations, when freed from fractions and surds.

In these general definitions, the straight line is supposed to be comprehended as it is the locus of simple equations. The loci of quadratic equations are shewn to be the conic sections, which are hence called lines of the second order, &c.

It is sufficiently plain from the nature of an equation, containing two variable quantities, that it must determine the position of every point of the curve defined by it, in the manner now described: for if any particular value of one of the variable quantities as x be assumed, the equation will then have one unknown quantity only, and being resolved, will give a precise number of corresponding values of y , which determine so many points of the curve.

As every point of the locus of an equation has the same general property, it must be one curve only, and from this equation all its properties may be derived. It is plain also that any curve line defined from the motion of a point according to a fixed rule, must either return into itself, or be extended *ad infinitum* with a continued curvature.

The equation however is supposed to be irreducible, because if it is not, the locus will be a combination of inferior lines: but this combination will possess the general properties of the lines of the order of the given equation.

It is to be observed all along, that the positive values of the ordinate as PM being taken upwards, the negative Pm will be placed downward on the opposite side of the base: and if positive values of the absciss as AP be assumed to the right from its beginning, the negative values, AP will be upon the left, and from these the points of the curve M , m , on that side are to be determined.

In

In the general definition of curves it is usual to suppose the co-ordinates to be at right angles. If the locus of any equation be described, and if the absciss be assumed on another base, and the ordinate be placed at a different angle, the new equation expressing their relation, tho' of a different form, will be of the same order as the original equation; and likewise will have in common with it, those properties which distinguish the equations of that particular curve.

This method of defining curves by equations may not be the fittest for a full investigation of the properties of a particular curve; but as their number is without limit, such a minute enquiry concerning all would be not only useless, but impossible. It has this great advantage however, that many of the general affections of all curves, and of the distinct orders, and also some of the most useful properties of particular curves may be easily derived from it.

I. The Determination of the Figure of a Curve from its Equation.

The general figure of the curve may be found by substituting successively particular values of x the absciss, and finding by the resolution of these equations the corresponding values of y the ordinate, and of consequence so many points of the curve. If numeral values be substituted for x , and also certain numbers for the known letters, the resolution of the equation gives numeral expressions of the ordinates; and from these by means of scales a mechanical description of the curve will be obtained, which may often be useful, both in pointing out the general disposition of the figure, and also in the practical applications of geometry.

Some more general suppositions may be of use in determining the figure; but these can be suggested only from the particular form of the equation in view. By supposing x to have certain relations to the known quantities, the values of y may become more simple, and the equation may be reduced to such a form as to shew the direction of the curve, and some of its obvious properties.

The following general observations also may be laid down.

1. If in any case a value of y vanishes, then the curve meets the base in a point determined by the corresponding value of x . Hence by putting $y=0$, the roots of the equation, which in that situation are values of x , will give the distances on the base at which the curve meets it.

2. If at a particular value of x , y becomes infinite, the curve
has

has an infinite arc, and the ordinate at that point becomes an asymptote.

3. If when x becomes infinitely great y vanishes, the base becomes an asymptote.

4. If any values of y become impossible, then so many intersections of the ordinate and curve vanish. If at any value of x all the values of y become impossible, the ordinate does not there meet the curve.

5. If two values of y become equal and have the same sign, the ordinate in that situation either touches the curve; or passes thro' an intersection of two of its branches, which is called a *punctum duplex*, or thro' an oval become infinitely little, called a *punctum conjugatum*.

In like manner is a *punctum triplex*, &c. to be determined.

The following example will illustrate this doctrine.

Let the equation be $ay^2 - xy^2 = x^3 + bx^2$: therefore $y^2 = \frac{x^3 + bx^2}{a - x}$ and $y = \pm \sqrt{\frac{x^3 + bx^2}{a - x}} = \pm x \sqrt{\frac{x + b}{a - x}}$.

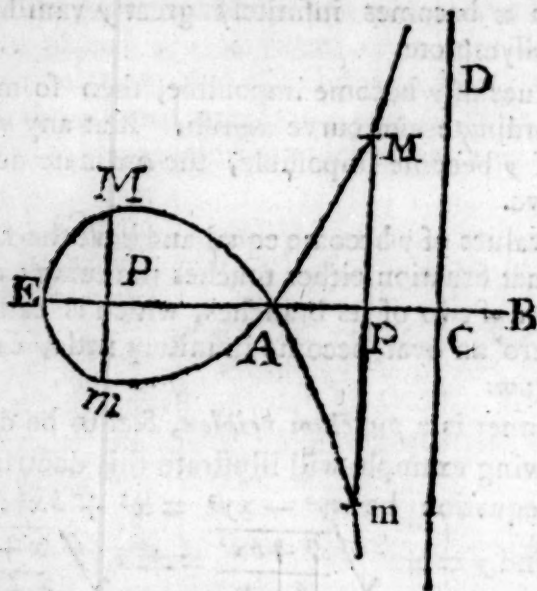
Let AB be assumed as a base on which the abscissæ are to be taken from A, and the ordinates perpendicular to it.

Since the two values of y are equal, but have opposite signs; PM, and Pm which represent them, must be taken equal to each other on opposite sides of AB; and it is plain that the parts of the curve on the two sides of AB, must be every way similar and equal.

If x is made equal to a , then $y = x \sqrt{\frac{x + b}{0}}$ which is an algebraical expression for infinity; therefore if AC is taken equal to a , the perpendicular CD will become an asymptote to the curve, which will have two infinite arcs. (Obs. 2.) If x is greater than a , the quantity under the radical sign becomes negative, and the values of y are impossible; that is no part of the curve lies beyond CD. (4.)

Both branches of the curve pass through A since $y = 0$, when $x = 0$. (1.) Let x be negative and $y = \mp x \sqrt{\frac{b - x}{a + x}}$; the values of y will be possible if x is not greater than b ; but if $x = b$, then $y = 0$, and if x is greater than b , the values of y become impossible; that is, if the abscissæ AP be taken to the left of A, and less than b , there will be two real equal values of y , viz PM, Pm on

on the opposite sides as before: if AE be taken equal to b , the curve will pass thro' E , and no part of it is beyond E . (1. and 4.)



The portion between A and E is called a *Nodus*.

If y be put $= 0$, then the values of x are $0, 0, -b$. That is the curve passes twice thro' A , or A is a punctum duplex, and it passes also thro' E as before. (1.)

In a similar method the locus of any other equation is to be traced: thus by varying the former equation, the figure of its locus will be varied. If $b=0$, then the points A and E coincide, the nodus vanishes, and A is called a *Cuspis*.

If b is negative, then E is to the right of A , which will now be a punctum conjugatum. The rest of the curve will be between E and C , and CD becomes an asymptote.

If $a=0$ then $-xy^2 = x^3 - bx^2$ or $y^2 = bx - x^2$ which is an equation to the circle of which $b = AE$ is the diameter.

II. General Properties of Curves from their Equations.

The general properties of equations lead to the general affections of curve lines. For example,

A straight line may meet a curve in as many points as there are units in the dimension of its equation; for so many roots may that equation have. An asymptote may cut a curve line in as many points excepting two as it has dimensions, and no more. The same may be observed of the tangent.

Impossible roots enter an equation by pairs, therefore the inter-

tersections of the ordinate and curve must vanish by pairs.

The curves of which the number expressing the order is odd, must have at least two infinite arcs; for the absciss may be so assumed that for every value of it, either positive or negative, there must be at least one value of y , &c.

The properties of the coefficients of the terms of equations mentioned Part II. Chap. I. furnish a great number of the curious and universal properties of curve lines. For example, the second term of an equation is the sum of the roots with the signs changed, and if the second term is wanting, the positive and negative roots must be equal. From this it is easy to demonstrate, "That if each of two parallel straight lines meet a curve line in as many points as it has dimensions, and if a straight line cut these two parallels, so that the sum of the segments of each on one side, be equal to the sum of the segments on the other, this straight line will cut any other line parallel to these in the same manner." Analogous properties, with many other consequences from them, may be deduced from the composition of the coefficients of the other terms.

Many properties of a particular order of curves may be inferred from the properties of equations of that order. Thus, "If a straight line cut a curve of the third order in three points, and if another straight line be drawn making a given angle with the former, and cutting the curve also in three points, the parallelepiped by the segments of one of these lines between its intersection with the other, and the points where it meets the curve, will be to the parallelepiped by the like segments of the other line, in a given ratio."—This depends upon the composition of the absolute term, and may be extended to curves of any order.

III. *The Subdivision of Curves.*

As lines are divided into orders from the dimensions of their equations, in like manner from the varieties of the equations of any order, may the different *Genera* and *Species* of that order be defined, and their peculiar properties discovered.

For this purpose a complete general equation is assumed of that order, and the varieties in the terms and coefficients which affect the figure of the locus are enumerated.

It was formerly observed, that the equations belonging to any one curve, may be of various forms, according to the position of the base and the angle which the ordinate makes with it, tho' they be all of the same order, and have certain properties, which distinguish them from the other equations of that order.

The locus of simple equations is a straight line. There are three species of lines of the second order, which are easily shewn to be the conic sections, reckoning the circle and ellipse to be one. *Seventy-eight* species have been numbered of the third order: and as the superior orders become too numerous to be particularly reckoned, it is usual only to divide them into certain general classes.

A complete general arrangement of the curves of any order, would furnish canons, by which the species of a curve whose equation is of that order might be found.

IV. *Of the place of Curves defined from other Principles, in the Algebraical System.*

If a curve line be defined from the section of a solid, or from any rule different from what has been here supposed, an equation to it may be derived, by which its order and species in the general system may be found. And for this purpose any base and any angle of the co-ordinates may be assumed, from which the equation may be most easily derived, or may be of the most simple form.

The three *Conic Sections* are of the second order, as their equations are universally quadratic; the *Cissoïd* of the ancients is of the third order, and the forty-second species, according to Sir *Isaac Newton's* enumeration; this is the curve defined by the equation in page 88, when $b=0$. The curve delineated in that page is the 41st species. When b is negative in that equation, the locus is the 43d species. The *Conchoid* of *Nicomædes* is of the fourth order; the *Cassinian* curve is also of the fourth order, &c.

It is to be observed, that not only the first definition of a curve may be expressed by an equation, but likewise any of those theorems called loci, in which some property is demonstrated to belong to every point of the curve. The expression of these propositions by equations, is sometimes difficult: no general rules can be given, and it must be left to the skill and experience of the learner.

S C H O L I U M.

This method of treating curve lines by equations, besides the uses already hinted at, has many others, which do not belong to this place: such are, the finding the tangents of curves, their curvature, their areas and lengths, &c. The solution of these problems has been accomplished by means of the equations to curves, tho' by employing concerning them a method of reasoning, different from what has been here explained.

CHAP.

C H A P. III.

I. *Construction of the Loci of Equations.*

THE description of a curve, according to the definition of it, is assumed in geometry as a *Postulate*.*

If the properties of a particular curve are investigated, it will appear that it may be described from a variety of data different from those assumed in the postulate, by demonstrating the dependence of the former upon the latter.

As the definitions of a curve may be various, so also may be the postulates, and a definition is frequently chosen from the mode of description connected with it: the simplicity of it, and even the accuracy with which it can be executed mechanically, have been considered as important circumstances.

Tho' an equation to a curve may be considered as a datum sufficient for its description, yet it is seldom possible to execute this description mechanically, by such a motion as is supposed in the definition by that equation. Hence other more simple motions are investigated from the equation, by which the curve defined by it may be described.

In the straight line, the circle, the conic sections, and a few curves of the higher orders, the most convenient postulates are generally understood. Hence it becomes useful to construct the locus of an equation to any of these lines; that is, to find from the equation, those data which from the established theory of the curve, are considered as necessary to its description in the postulate, or in any problem founded upon it.

An example of such a construction is in the problem of the next section.

II.

* A postulate in geometry seems to be improperly called a mechanical principle. No geometrical line whatever can be described mechanically according to the mathematical definition; and therefore in theory, the solutions of problems by the conic sections, or by any of the higher orders, is to be considered as equally perfect with those by the circle and straight lines. When geometrical operations are used in practice, then those lines of whatever order and those methods of describing them are to be preferred, by which the construction required may be most easily and accurately performed. There are but few mechanical operations which admit of tolerable accuracy, and hence the great advantage of arithmetical calculations in the practical arts founded on geometry. By these the more complicated constructions of geometry are reduced to those simple operations which are found by experience to be capable of greatest exactness.

II. Solution of Problems.

The solution of geometrical problems by algebra is much promoted by describing the loci of the equations arising from these problems.

For this purpose equations are to be derived, according to the methods formerly described, and then to be reduced to two, containing each the same two unknown quantities. The loci of these equations are to be described, the two unknown quantities being considered as the co-ordinates, and placed at the same angle in both. The co-ordinates at an intersection of the loci will be common to both, and give a solution of the problem.

The simplicity of a construction obtained by this method will depend upon a proper notation, and the choice of the equations of which the loci are to be described. These will frequently be different from what would be proper in a different method of solution.

P R O B. III.

To find a Point F in the Base of the given Triangle ABC, so that the Sum of the Squares of FE, FD drawn from it perpendicular upon the two Sides, may be equal to a given Space.

Draw BH, CG perpendicular on the two sides, and let $FD = x$, $FE = y$, $BF = z$, $BC = b$, $BH = p$, $CG = r$, and the given space $FD^2 + FE^2 = m^2$.

From similar triangles $z : x ::$

$$b : r \text{ and } z = \frac{bx}{r}$$

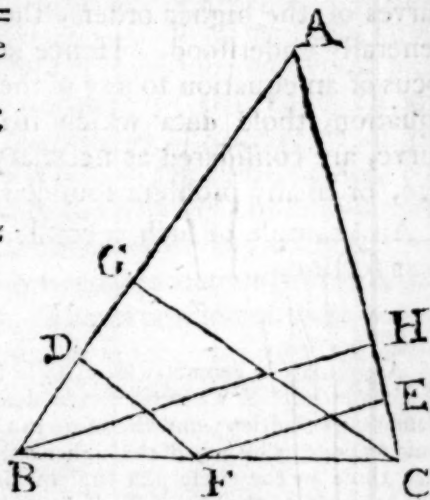
$$\text{Also } b - z : y :: b : p, \text{ and } z = b - \frac{yb}{p}$$

$$\text{therefore } \frac{bx}{r} = b - \frac{yb}{p}$$

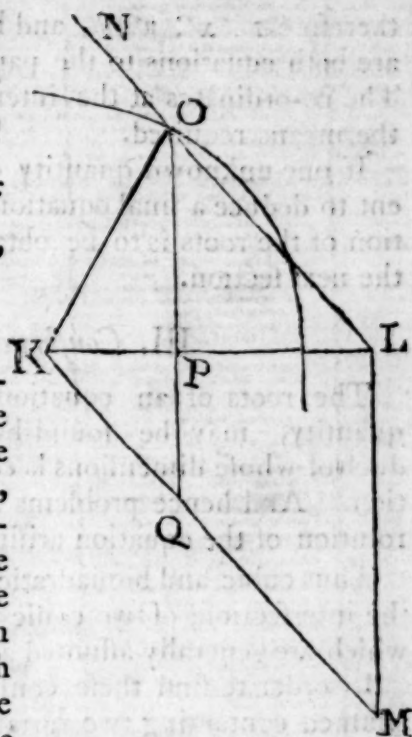
That is $y = p - \frac{px}{r}$, an equation to a straight line.

But $x^2 + y^2 = m^2$ of which the locus is a circle having m for the radius. By constructing these loci, their intersection will give a solution of the problem.

Let $KL = CG (= r)$ be at right angles to $LM = BH (= p)$ join



join KM to which let LN be parallel; LN is the locus of the equation $y = p - \frac{px}{r}$; for let any line OPQ be drawn parallel to LM, if $KP = x$ then $PQ = \frac{px}{r}$, and $QO = LM = p$, therefore $PO = y = p - \frac{px}{r}$.



About the center K, with a distance equal to the line m , let a circle be described; that circle will be the locus of the equation $m^2 = x^2 + y^2$, for it is plain that if OP be any perpendicular from the circumference upon KL, KP being x , OP will be y . Either of the points therefore in which these two loci intersect each other, as O, will give OP an ordinate in both equations, KP being the common absciss; therefore KP, OP are the two perpendiculars required, from which the point F is easily found.

The construction might have been made on figure 1st, with fewer lines. If the circle touches LN there is only one solution which is a minimum, and if the circle does not meet LN, the problem becomes impossible.

When the circle touches LN, the radius m must be equal to the perpendicular from K on LN, or from L on KM. This perpendicular is equal to $\frac{pr}{\sqrt{p^2 + r^2}}$ or a fourth proportional to MK,

KL, and LM, and its square therefore is the least sum of the squares of the perpendiculars from a point in the base on the two sides.

It may be remarked also, that the point which gives the sum of the squares a minimum, is found by dividing the base, in the proportion of the squares of the two sides of the triangle; and this is easily demonstrated from the preceeding construction.

P R O B. IV.

Between two given Lines to find two mean Proportionals.

Let the lines be a and b , and let the two means be x and y ;
therefore

therefore $a : x : y : b$, and hence $ay = x^2$, and $bx = y^2$, which are both equations to the parabola, and are easily constructed. The co-ordinates at the intersection of these two loci, will be the means required.

If one unknown quantity only is assumed, or if it is convenient to deduce a final equation containing only one, the construction of the roots is to be obtained by the method mentioned in the next section.

III. *Construction of Equations.*

The roots of an equation, containing only one unknown quantity, may be found by the intersection of lines, the product of whose dimensions is equal to the dimension of that equation. And hence problems are resolved without an algebraical solution of the equation arising from them.

Thus cubic and biquadratic equations may be constructed by the intersections of two conic sections as the circle and parabola, which are generally assumed as being most easily described.

In order to find these constructions, a new equation is to be assumed containing two variable quantities, one of which is the unknown quantity of the given equation, and the other by substitution is to be inserted also in the given equation; the intersection of the loci of these equations will exhibit the roots required.

Canons may be derived for the construction of particular orders, without assuming the new equation.

The final equation from prob. 4. would be $x^3 = a^2b$, which being constructed according to the rules, exhibits the common geometrical solution of that problem by the circle and parabola.

If an equation be assumed as $ay = x^2$, the other by substitution becomes $xy = ab$; the locus of the former is a parabola, and of the latter an hyperbola, one of its asymptotes being the base, and the co-ordinates at their intersection will represent x and y ; The first of the two means is x , and in this case y is the other.

Equations also might be assumed so as to give a solution of this problem by other combinations of two of the conic sections, one of them not being the circle.

As geometrical magnitudes may be represented by algebra, so algebraical quantities and numbers may be represented by lines. Hence this construction of equations has sometimes been used as an easy method of approximation to a certain degree of accuracy: but as it depends upon mechanical operations, it cannot be continued at pleasure.

S C H O.

S C H O L I U M.

If the relation between the ordinate and absciss be fixed, but not expressible by a finite equation, the curve is called *Mechanical*. This class is also sometimes defined by equations, by supposing either x or y in a finite equation, to be a curve line, of which the relation to a straight line cannot be expressed in finite terms.

If the variable quantities x or y enter the exponents of any term of an equation, the locus of that equation is called an *Exponential Curve*.

Many properties of these two kinds of curves may be discovered from their equations,

THE END.

E R R A T A.

Page	Line	For	Read
4	16	dividing	adding
9	7	$a-b$)	$3a-b$)
14	6 bott.	$+b$	$+b^2$
16	8	$)a^3$	$)a^2$
18	10	$=c$	$=bc$
20	28	$=12x$	$=24x$
20	last	if x	if $3x$
23	20	$=y$	$=3y$
24	20	$\frac{16z-3700}{6}$	$\frac{16z-3700}{9}$
33	11	$=a^3$	$=a^2$
42	3	$b+y$	$\frac{b+y}{2}$
42	10	$26y^2$	$2by^2$
44	3 after signs	insert a comma	
44	3 dele the comma	after quadratie.	
45	5 bott.	$-\frac{a}{4}$	$-\frac{a}{2}$
47	21	$=\frac{a^2}{2}$	$=-\frac{a^2}{2}$
54	6 dele the		
55	21	$br \quad cr^2$	$br + cr^2$
55	7 bott.	by $\frac{r-1}{d \times r + 1}$	by that aliquot part.
56	12	$d \times r + 1$	$\frac{d \times r^3 + 1}{x}$
63	13	$x=889$	$x=8.885$ nearly

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